

1. Show that the inversion $z \mapsto \frac{1}{\bar{z}}$ of the complex plane $\mathbb{C}$ corresponds to a rotation of π about the X -axis of the Riemann sphere under stereographic projection.

2. Evaluate the following integrals.

(a) $\int_{|z|=1} \frac{e^z}{z^m} dz$ for every integer m

(b) $\int_{|z|=1} \frac{1}{z^2(z^2 - 4)e^z} dz$

(c) $\int_{|z|=1} \frac{\cosh(z)}{z^3} dz$

3. Determine the image of the horizontal line through i under the fractional linear transformation that interchanges 0 and 1 and maps -1 to $1 + i$.

4. Let $f(z) = \frac{1}{z(z-1)(z-2)}$.

Determine the Laurent expansion of $f(z)$ on each of the following annuli:

(a) $0 < |z| < 1$

(b) $1 < |z| < 2$

(c) $|z| > 2$ (that is, $2 < |z| < +\infty$)

5. $g(z) = e^{\frac{1}{z}}$ has an essential singularity at 0.

(a) Compute the residue of $g(z)$ at 0.

(b) Evaluate the integral $\oint_{|z|=2} g(z) dz$.

6. Evaluate $\oint_{|z|=2} \frac{z}{\cos(z)} dz$.

7. Suppose $P(z)$ and $Q(z)$ are polynomials with $\deg(P) < \deg(Q)$ and such that the zeros of $Q(z)$ are simple zeros at the points $z_1, \dots, z_N$. Show that the partial fraction decomposition of $f(z) = \frac{P(z)}{Q(z)}$ is given by

$$f(z) = \sum_{k=1}^N \frac{P(z_k)}{Q'(z_k)} \cdot \frac{1}{z - z_k}.$$

8. Given $a > 0$, show that $\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^4 + 1} dx = \frac{\pi}{\sqrt{2}} e^{-\frac{a}{\sqrt{2}}} \left[\cos\left(\frac{a}{\sqrt{2}}\right) + \sin\left(\frac{a}{\sqrt{2}}\right) \right]$.

9. Show that $\int_0^{2\pi} \frac{1}{a + b \sin(\theta)} d\theta = \frac{2\pi}{\sqrt{a^2 - b^2}}$, for $0 < b < a$.

10. Show that $\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(\theta) + r^2} d\theta = 1$, for $0 \leq r < 1$.

[Remark: The integrand $\frac{1-r^2}{1-2r \cos(\theta)+r^2}$ is called the Poisson kernel.]