Math 142B Homework Assignment 5
Due 11:00pm Thursday, March 14, 2023

1. For $n = 0, 1, 2, 3, \ldots$ let $a_n = \left[\dfrac{4 + 2(-1)^n}{5}\right]^n$.

   (a) Find

      i. $\limsup (a_n)^{\frac{1}{n}}$

      ii. $\liminf (a_n)^{\frac{1}{n}}$

      iii. $\limsup \left|\dfrac{a_{n+1}}{a_n}\right|$

      iv. $\liminf \left|\dfrac{a_{n+1}}{a_n}\right|$

   (b) Do the series $\sum a_n$ and $\sum (-1)^n a_n$ converge? Briefly justify your answers.

   (c) Find the radius of convergence and exact interval of convergence of the power series $\sum a_n x^n$ with a_n as above.

2. Let $\sum a_n x^n$ be a power series with radius of convergence R. Prove:

   (a) If all the coefficients a_n are integers and $a_n \neq 0$ for infinitely many n, then $R \leq 1$.

   (b) If $\limsup |a_n| > 0$, then $R \leq 1$.

3. (a) Suppose $\sum a_n x^n$ has finite radius of convergence R and $a_n \geq 0$ for all n. Show that if the series converges at $x = R$, then it also converges at $x = -R$.

   (b) Exhibit an example of a power series whose interval of convergence is exactly $(-1, 1]$. (**Note:** "Exhibit" means "Show that the example has the required properties.")

4. (a) Verify that $e^{-x^2} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$ for all $x \in \mathbb{R}$, since $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for all $x \in \mathbb{R}$.

   (b) Write $F(x) = \displaystyle\int_0^x e^{-t^2}\, dt$ as a power series. Be sure to briefly explain how you know that the power series for $F(x)$ converges for all $x \in \mathbb{R}$.

5. Let $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for $x \in \mathbb{R}$. Using only the properties of power series, show that $f' = f$.

6. For $x \in \mathbb{R}$, let

$$s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

   Prove:

   (a) $s' = c$ and $c' = -s$.

   (b) $\left(s^2 + c^2\right)' = 0$.

   (c) $s^2 + c^2 = 1$.

7. (a) Show that $\sum_{n=0}^{\infty}(-1)^n x^{2n} = \dfrac{1}{1+x^2}$ for $x \in (-1,1)$.

   (b) Show that $\arctan(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{2n+1} x^{2n+1}$ for $x \in (-1,1)$.

   (c) Show that the equality in (b) also holds for $x = 1$. Use this to find a fun formula for π.

   (d) What happens at $x = -1$?

8. Prove the following *comparison tests.* Let f and g be continuous functions on (a,b) such that $0 \leq f(x) \leq g(x)$ for all $x \in (a,b)$ and where a could be $-\infty$ and b could be $+\infty$.

   (a) If $\int_a^b g(x)\,dx < \infty$, then $\int_a^b f(x)\,dx < \infty$.

   (b) If $\int_a^b f(x)\,dx = +\infty$, then $\int_a^b g(x)\,dx = +\infty$.

9. (a) Using a comparison test, show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx < \infty$.

   (b) Show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

10. Suppose f is continuous on (a,b), where a could be $-\infty$ and b could be $+\infty$. Show that if $\int_a^b |f(x)|\,dx < \infty$, then the integral $\int_a^b f(x)\,dx$ exists and is finite.