

Math 142B Homework Assignment 3
Due 11:00pm Thursday, February 15, 2024

1. Let $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$
- (a) Compute the upper and lower Darboux integrals for f on the interval $[0, b]$.
- (b) Is f integrable on $[0, b]$? Be sure to justify your answer.
2. Let f be a bounded function on $[a, b]$. Suppose there exist sequences (L_n) and (U_n) of upper and lower Darboux sums for f such that $\lim(U_n - L_n) = 0$.

Show that f is integrable on $[a, b]$ and that $\int_a^b f = \lim L_n = \lim U_n$.

3. Let f be integrable on $[a, b]$, and suppose g is a function on $[a, b]$ such that $g(x) = f(x)$ except for finitely many $x \in [a, b]$.

Show that g is integrable on $[a, b]$ and that $\int_a^b g = \int_a^b f$.

4. Show that if f is integrable on $[a, b]$, then f is integrable on every interval $[c, d] \subseteq [a, b]$
5. Show that a decreasing function f on $[a, b]$ is integrable.
6. Exhibit an example of a function f on $[0, 1]$ that is *not* integrable but for which $|f|$ is integrable.
7. Let f be a bounded function on $[a, b]$ so that there is $B > 0$ for which $|f(x)| \leq B$ for all $x \in [a, b]$.

- (a) Show that

$$U(f^2, P) - L(f^2, P) \leq 2B [U(f, P) - L(f, P)]$$

for all partitions P of $[a, b]$.

- (b) Show that if f is integrable on $[a, b]$, then f^2 is also integrable on $[a, b]$.
8. Let f and g be integrable functions on $[a, b]$.
- (a) Show that fg is integrable on $[a, b]$.
- (b) Show that $\max(f, g)$ and $\min(f, g)$ are integrable on $[a, b]$.
9. Suppose f and g are continuous functions on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Prove that there exists $x \in (a, b)$ at which $f(x) = g(x)$.
10. (a) Prove that if f and g are continuous functions on $[a, b]$ with $g(t) \geq 0$ for all $t \in [a, b]$, then there exists $x \in (a, b)$ such that

$$\int_a^b f(t)g(t) dt = f(x) \int_a^b g(t) dt.$$

- (b) Show that the *Intermediate Value Theorem for Integrals* is a special case of part (a).
- (c) Does the conclusion in part (a) hold if $[a, b] = [-1, 1]$ and $f(t) = g(t) = t$ for all $t \in [a, b]$? Be sure to justify your answer.