Math 142B Homework Assignment 2
Due 11:00pm Thursday, February 1, 2024

1. Suppose f is differentiable on $\mathbb{R}$, $1 \le f'(x) \le 2$ for all $x \in \mathbb{R}$, and $f(0) = 0$. Show that $x \le f(x) \le 2x$ for all $x \ge 0$.

2. Define $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ by $f(x) = \tan(x)$ and let $g : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be the the inverse of f. Determine a formula for $g'(y)$ for $y \in \mathbb{R}$. (See Exercise 29.16 in your text.)

3. Let f be differentiable on $\mathbb{R}$ with $d = \sup\{|f'(x)| \mid x \in \mathbb{R}\} < 1$.
   (a) Select $s_0 \in \mathbb{R}$ and define $s_n = f(s_{n-1})$ for $n \ge 1$. Show that (s_n) is a convergent sequence. (See Exercise 29.18 in your text.)
   [Hint: Show (s_n) is Cauchy by first showing that $|s_{n+1} - s_n| \le d\,|s_n - s_{n-1}|$ for $n \ge 1$.]
   (b) Show that f has a *fixed point.* That is, show that $f(x_0) = x_0$ for some $x_0 \in \mathbb{R}$.

4. Let f be a function defined on some open interval $(0, a)$, and define $g(y) = f\left(\frac{1}{y}\right)$ for $y \in \left(\frac{1}{a}, \infty\right)$ with $\frac{1}{a} := 0$ when $a = \infty$.
   Show that $\lim_{x \to 0^+} f(x)$ exists if and only if $\lim_{y \to \infty} g(y)$ exists, in which case the limits are equal.

5. Let f be differentiable on some interval (c, ∞) such that $\lim_{x\to\infty} \left[f(x) + f'(x)\right] = L$, with L finite.
   Show that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = 0$. $\left[\text{Hint: Write } f(x) = \frac{f(x)\ e^x}{e^x}.\right]$

6. For $x \in \mathbb{R}$, let
$$f(x) = x + \cos(x)\sin(x) \quad \text{and} \quad g(x) = e^{\sin(x)}\left[x + \cos(x)\sin(x)\right].$$
   (a) Show that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = +\infty$.
   (b) Show that $f'(x) = 2\left[\cos(x)\right]^2$ and $g'(x) = e^{\sin(x)}\cos(x)\left[2\cos(x) + f(x)\right]$.
   (c) Show that $\dfrac{f'(x)}{g'(x)} = \dfrac{2\,e^{-\sin(x)}\cos(x)}{2\cos(x) + f(x)}$ if $\cos(x) \ne 0$ and $x > 3$.
   (d) Show that $\lim_{x\to\infty} \dfrac{2e^{-\sin(x)}\cos(x)}{2\cos(x) + f(x)} = 0$ and yet $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$ is *not* defined.

7. Find the Taylor series for $\cos(x)$ and indicate why it converges to $\cos(x)$ for all $x \in \mathbb{R}$.

8. Let $g(x) = \begin{cases} 0 & \text{if } x = 0, \\ e^{-1/x^2} & \text{otherwise.} \end{cases}$
   (a) Show that $g^{(n)}(0)$ for all $n \in \mathbb{N}$.
   (b) Show that the Taylor series for g about 0 agrees with g only at $x = 0$.

9. Prove that $|\sin(x + h) - (\sin(x) + h\cos(x))| \le \dfrac{h^2}{2}$ for every pair of real numbers x and h.

10. Suppose f is differentiable on (a, b), f' is bounded on (a, b), f' never vanishes on (a, b), and the sequence (x_n) in (a, b) converges to $\bar{x} \in (a, b)$.
    Show that if $x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})}$ for all $n \ge 0$, then $f(\bar{x}) = 0$.