Math 142A Homework Assignment 4
Due Tuesday, February 27, 2024

1. Show that if $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers, then $\sum \sqrt{a_n\, b_n}$ converges.

2. Find a series $\sum a_n$ which diverges by the Root Test but for which the Ratio Test gives no information.

3. Let (a_n) be a sequence of nonzero real numbers such that the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ is a constant sequence. Show that $\sum a_n$ is a geometric series.

4. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $\liminf |a_n| = 0$. Prove there is a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $\displaystyle\sum_{k=1}^{\infty} a_{n_k}$ converges.

5. (a) Exhibit an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.
   (b) Show that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ also converges.
   (c) Exhibit an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

6. Prove that if (a_n) is a decreasing sequence of positive real numbers and if $\sum a_n$ converges, then $\lim\limits_{n\to\infty} n a_n = 0$

7. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous at the point x_0 and that $f(x_0) > 0$. Show that there is an interval $I_n = \left(x_0 - \frac{1}{n},\ x_0 + \frac{1}{n}\right)$ for some $n \in \mathbb{N}$ for which $f(x) > 0$ for every $x \in I_n$.

8. A function $f : D \to \mathbb{R}$ is said to be *Lipschitz* provided that there is a number $C \geq 0$ with

$$|f(u) - f(v)| \leq C\,|u - v| \quad \text{for all } u \text{ and } v \text{ in } D.$$

   Show that a Lipschitz function is continuous.

9. Suppose the function $\lambda : \mathbb{R} \to \mathbb{R}$ has the the property that $\lambda(u+v) = \lambda(u) + \lambda(v)$ for all u, v.
   (a) Define the number m by $m := \lambda(1)$. Show that $\lambda(x) = m\,x$ for all rational numbers x.
   (b) Show that if λ is continuous, then $\lambda(x) = m\,x$ for all $x \in \mathbb{R}$.

10. Let f be a real-valued function whose domain is a subset of $\mathbb{R}$.

    Show that f is continuous at $x_0 \in \operatorname{dom}(f)$ if and only if for every sequence (x_n) in $\operatorname{dom}(f)\setminus\{x_0\}$ converging to x_0, we have $\lim\limits_{n\to\infty} f(x_n) = f(x_0)$.