Math 142A Homework Assignment 3
Due 11:00pm Tuesday, February 13, 2024

1. (a) Let (s_n) be a monotone sequence. Prove that (s_n) converges if and only if (s_n^2) converges.
   (b) Find a non-monotone sequence (t_n) such that (t_n^2) converges but (t_n) does not converge.

2. Let (r_n) be an enumeration of the set $\mathbb{Q}$ of all rational numbers. Show there exists a subsequence (r_{n_k}) such that $\lim_{k\to\infty} = +\infty$.

3. Prove that $\liminf s_n = -\limsup(-s_n)$ for every sequence (s_n).
   [Hint: See Definition 10.6 and Exercise 5.4]

4. Let S be a bounded set. Prove that there is an increasing sequence (s_n) in S such that $\lim s_n = \sup S$. Explain why if $\sup S$ is in S, then it suffices to define $s_n = \sup S$ for all n.

5. Show that a monotonically increasing sequence is bounded if it has a bounded subsequence.

6. Suppose the sequence (s_n) is monotonically increasing and that it has a convergent subsequence. Show that (s_n) converges.

7. Let (s_n) and (t_n) be bounded sequences of nonnegative real numbers. Prove that

$$\limsup s_n t_n \leq (\limsup s_n)(\limsup t_n).$$

8. Prove that (s_n) is bounded if and only if $\limsup |s_n| \in \mathbb{R}$ (that is, $\limsup |s_n| < +\infty$).

9. Let (s_n) be a bounded sequence of nonzero real numbers. Prove that

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n}.$$

10. Let (s_n) be a sequence of nonnegative numbers. For each n, define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. (σ_n) is called the sequence of Cesàro means for (s_n).
    (a) Show that $\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n$.
    (b) Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.
    (c) Exhibit an example for which $\lim \sigma_n$ exists, but $\lim s_n$ does not exist.