**Math 142A Homework Assignment 2**
**Due Tuesday, January 30, 2024**

1. Let $(t_n)$ be a bounded sequence; that is, there exists $M \geq 0$ such that $|t_n| \leq M$ for all $n$. Let $(s_n)$ be a sequence such that $\lim s_n = 0$. Prove that $\lim(s_n t_n) = 0$.

2. Consider three sequences $(a_n), (b_n)$, and $(s_n)$ such that $a_n \leq s_n \leq b_n$ for all $n$, and $\lim a_n = \lim b_n = s$. Prove that $\lim s_n = s$.

3. Suppose $(s_n)$ and $(t_n)$ are sequences such that $|s_n| \leq t_n$ for all $n$ and $\lim t_n = 0$. Prove that $\lim s_n = 0$.

4. Suppose that $s_n \neq 0$ for every index $n$ and that the limit $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ is defined.
   (a) Show that if $L < 1$, then $\lim s_n = 0$.
   (b) Show that if $L > 1$, then $\lim |s_n| = +\infty$.
   (See Exercise 9.12 in your text for a hint.)

5. Let $s_1 = 1$, and for $n \geq 1$, let $s_{n+1} = \sqrt{s_n + 1}$. It turns out that $(s_n)$ converges. Assume this fact and show that limit $\lim s_n = \frac{1}{2}\left(1 + \sqrt{5}\right)$.

6. Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$.
   (a) Show that if $a = \lim x_n$, then $a = \frac{1}{3}$ or $a = 0$.
   (b) Does $\lim x_n$ exist? Justify your answer.
   (c) Explain the apparent contradiction between the result in part (a) and part (b).

7. Show that $\displaystyle\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

8. (a) Verify that $1 + a + a^2 + \cdots + a^n = \dfrac{1 - a^{n+1}}{1 - a}$ for $a \neq 1$.
   (b) Determine $\displaystyle\lim_{n\to\infty} \left(1 + a + a^2 + \cdots + a^n\right)$ for $|a| < 1$.
   (c) What is $\displaystyle\lim_{n\to\infty} \left(1 + a + a^2 + \cdots + a^n\right)$ for $a \geq 1$?

9. Let $S$ be a bounded nonempty subset of $\mathbb{R}$ such that $\sup(S) \notin S$. Show that there is a sequence $(s_n)$ of points in $S$ such that $\lim s_n = \sup(S)$.

10. Let $(s_n)$ be a sequence such that $|s_{n+1} - s_n| < 2^{-n}$ for all $n \in \mathbb{N}$.
    (a) Prove that $(s_n)$ is a Cauchy sequence and, therefore, a convergent sequence.
    (b) Is $(s_n)$ a Cauchy sequence if we only assume that $|s_{n+1} - s_n| < \dfrac{1}{n}$ for all $n \in \mathbb{N}$?