Math 20E Homework Assignment 5
Due 11:00pm Tuesday, May 21, 2024

1. Compute the area enclosed by the ellipse $\left(\frac{x}{c}\right)^2 + \left(\frac{y}{d}\right)^2 = 1$.

2. Find the area of the region between the x-axis and the cycloid parametrized by $\mathbf{r}(t) = (t - \sin(t),\ 1 - \cos(t))$ with $0 \le t \le 2\pi$.

3. Let $P(x, y) = \dfrac{-y}{x^2 + y^2}$ and $Q(x, y) = \dfrac{x}{x^2 + y^2}$, and let D be the unit disk $D = \{(x, y) \mid x^2 + y^2 \le 1\}$.
   (a) Evaluate the area integral $\iint_D \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) dx\, dy$ over the unit disk D.
   (b) Evaluate the line integral $\int_{\partial D} P\, dx + Q\, dy$ around ∂D, the unit circle with positive orientation.
   (c) Briefly explain why Green's theorem failed.

4. Let $\mathbf{r}(x, y, z) = (x,\ y,\ z)$ and $r(x, y, z) = \sqrt{x^2 + y^2 + z^2} = \|\mathbf{r}\|$. Verify the following identities.
   (a) $\nabla\left(\dfrac{1}{r}\right) = -\dfrac{\mathbf{r}}{r^3}$.
   (b) $\boldsymbol{\nabla} \cdot \left(\dfrac{\mathbf{r}}{r^3}\right) = 0$.
   (c) $\boldsymbol{\nabla} \times \mathbf{r} = \mathbf{0}$.

5. Let C be the closed, piecewise smooth curve formed by traveling in straight lines between the points $(0, 0, 0)$, $(2, 1, 5)$, $(1, 1, 3)$, and back to $(0, 0, 0)$ in that order. Use Stokes' theorem to evaluate the line integral
$$\int_C (xyz)\, dx + (xy)\, dy + (x)\, dz.$$

6. Evaluate the surface integral $\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S}$, where S is the portion the surface of a sphere defined by $x^2 + y^2 + z^2 = 1$ and $x + y + z \ge 1$, and where $\mathbf{F} = \mathbf{r} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$, with $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

7. Let $\mathbf{F} = x^2\,\mathbf{i} + (2xy + x)\,\mathbf{j} + z\,\mathbf{k}$. Let C be the circle $x^2 + y^2 = 1$ and S the disk $x^2 + y^2 \le 1$ within the plane $z = 0$.
   (a) Determine the flux of $\mathbf{F}$ out of S.
   (b) Determine the circulation of $\mathbf{F}$ around C.
   (c) Find the flux of $\boldsymbol{\nabla} \times \mathbf{F}$. Verify Stokes' theorem directly in this case.

8. Let $\mathbf{F} = (0,\ -z,\ 1)$. Let S be the spherical cap $x^2 + y^2 + z^2 = 1$, where $z \ge \frac{1}{2}$.
   (a) Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ directly as a surface integral.
   (b) Verify that $\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{A}$, where $\mathbf{A} = (0,\ x,\ xz)$.
   (c) Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ using Stokes' theorem.

9. Let $\mathbf{F} = (y^2,\, x^2,\, z^2)$. Verify that $\displaystyle\int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r}$, for any two simple closed curves $\mathcal{C}_1, \mathcal{C}_2$ going around a cylinder whose central axis is the z-axis; that is, any cylinder whose equation is of the form $x^2 + y^2 = R^2$.