

Math 142B Homework Assignment 4
Due 11:00pm Thursday, May 23, 2024

- Consider $f_n(x) = nx^n(1-x)$ for $x \in [0, 1]$.
 - Find $f(x) = \lim_{n \rightarrow \infty} f_n(x)$.
 - Does $f_n \rightarrow f$ uniformly on $[0, 1]$? Be sure to justify your answer.
 - Does $\int_0^1 f_n(x) dx$ converge to $\int_0^1 f(x) dx$? Be sure to justify your answer.
- Prove that a sequence (f_n) of functions on a set $S \subseteq \mathbb{R}$ converges uniformly to a function f on S if and only if $\lim_{n \rightarrow \infty} \sup \{|f(x) - f_n(x)| \mid x \in S\} = 0$.
- Prove that if (f_n) is a sequence of functions uniformly continuous on an interval (a, b) and if $f_n \rightarrow f$ uniformly on (a, b) , then f is also uniformly continuous on (a, b) .
- Let (f_n) be a sequence of continuous functions on $[a, b]$ that converges uniformly to f on $[a, b]$. Show that if (x_n) is a sequence in $[a, b]$ with $x_n \rightarrow x$, then $f_n(x_n) \rightarrow f(x)$.
- Let (f_n) be a sequence of functions on a set $S \subset \mathbb{R}$ such that $f_n \rightarrow f$ uniformly on S . Prove that (f_n) is uniformly Cauchy on S .
- Let (f_n) be a sequence of bounded functions on a set S such that $f_n \rightarrow f$ uniformly on S . Prove that f is bounded on S .
- Show that if $\sum |a_k| < \infty$, then $\sum a_k x^k$ converges uniformly on $[-1, 1]$ to a continuous function.
 - Does $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ represent a continuous function on $[-1, 1]$?
- Let $0 < a < 1$. Show that the series $\sum_{n=0}^{\infty} x^n$ converges uniformly on $[-a, a]$ to $\frac{1}{1-x}$.
 - Does the series $\sum_{n=0}^{\infty} x^n$ converge uniformly on $(-1, 1)$ to $\frac{1}{1-x}$?
- Let (f_n) be a sequence of continuous functions on $[a, b]$ such that $(f_n(x))$ is an increasing sequence of real numbers for each $x \in [a, b]$. Prove that if $f_n \rightarrow f$ pointwise on $[a, b]$ and if f is continuous on $[a, b]$, then $f_n \rightarrow f$ uniformly on $[a, b]$. (This is called *Dini's theorem*.)
- For $n = 0, 1, 2, 3, \dots$ let $a_n = \left[\frac{4 + 2(-1)^n}{5} \right]^n$.
 - Find
 - $\limsup (a_n)^{\frac{1}{n}}$
 - $\liminf (a_n)^{\frac{1}{n}}$
 - $\limsup \left| \frac{a_{n+1}}{a_n} \right|$
 - $\liminf \left| \frac{a_{n+1}}{a_n} \right|$
 - Do the series $\sum a_n$ and $\sum (-1)^n a_n$ converge? Briefly justify your answers.
 - Find the radius of convergence and exact interval of convergence of the power series $\sum a_n x^n$ with a_n as above.