Math 142B Homework Assignment 3
Due 11:00pm Thursday, May 9, 2024

1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = \begin{cases} 0 & \text{if } t < 0, \\ t & \text{if } 0 \le t \le 1, \\ 4 & \text{if } t > 1. \end{cases}$

   (a) Determine the function $F(x) = \int_0^x f(t)\,dt$.

   (b) Where is F continuous?

   (c) Where is F differentiable? Compute $F'(x)$ at the points x where F is differentiable.

2. Let f be a continuous function on $\mathbb{R}$ and define $F(x) = \int_{x-1}^{x+1} f(t)\,dt$ for $x \in \mathbb{R}$.

   Show that F is differentiable on $\mathbb{R}$ and compute F'.

3. Let f be a continuous function on $\mathbb{R}$ and define $G(x) = \int_0^{\sin(x)} f(t)\,dt$ for $x \in \mathbb{R}$.

   Show that G is differentiable on $\mathbb{R}$ and compute G'.

4. Suppose f is a continuous function on $[a, b]$. Show that if $\int_a^b f(x)^2\,dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

5. Show that if f is a continuous real-valued function on $[a, b]$ satisfying $\int\limits_a^b f(x)g(x)\,dx = 0$ for every continuous function g on $[a, b]$, then $f(x) = 0$ for all $x \in [a, b]$.

6. (a) Show that

      i. $\int_0^1 x^{-p}\,dx = \frac{1}{1-p}$ if $0 < p < 1$.

      ii. $\int_0^1 x^{-p}\,dx = +\infty$ if $p > 1$.

   (b) Show that $\int_0^\infty x^{-p}\,dx = +\infty$ for all $p > 0$.

7. Compute

   (a) $\int_0^1 \log(x)\,dx$

   (b) $\int_2^\infty \frac{\log(x)}{x}\,dx$

   (c) $\int_0^\infty \frac{1}{1+x^2}\,dx$

8. Prove the following *comparison tests*. Let f and g be continuous functions on (a, b) such that $0 \le f(x) \le g(x)$ for all $x \in (a, b)$ and where a could be $-\infty$ and b could be $+\infty$.

   (a) If $\int_a^b g(x)\,dx < \infty$, then $\int_a^b f(x)\,dx < \infty$.

   (b) If $\int_a^b f(x)\,dx = +\infty$, then $\int_a^b g(x)\,dx = +\infty$.

9. (a) Using a comparison test, show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx < \infty$.

   (b) Show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

10. Suppose f is continuous on (a,b), where a could be $-\infty$ and b could be $+\infty$. Show that if $\int_a^b |f(x)|\,dx < \infty$, then the integral $\int_a^b f(x)\,dx$ exists and is finite.