Math 142B Homework Assignment 4
Due 11:00pm Wednesday, March 1, 2023

1. Let f be a continuous function on $\mathbb{R}$ and define $F(x) = \int_{x-1}^{x+1} f(t)\,dt$ for $x \in \mathbb{R}$.

   Show that F is differentiable on $\mathbb{R}$ and compute F'.

2. Consider $f_n(x) = nx^n(1-x)$ for $x \in [0,1]$.
   (a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.
   (b) Does $f_n \to f$ uniformly on $[0,1]$? Be sure to justify your answer.
   (c) Does $\int_0^1 f_n(x)\,dx$ converge to $\int_0^1 f(x)\,dx$? Be sure to justify your answer.

3. Prove that a sequence (f_n) of functions on a set $S \subseteq \mathbb{R}$ converges uniformly to a function f on S if and only if $\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| \mid x \in S\} = 0$.

4. Prove that if (f_n) is a sequence of functions uniformly continuous on an interval (a,b) and if $f_n \to f$ uniformly on (a,b), then f is also uniformly continuous on (a,b).

5. Let (f_n) be a sequence of continuous functions on $[a,b]$ that converges uniformly to f on $[a,b]$. Show that if (x_n) is a sequence in $[a,b]$ with $x_n \to x$, then $f_n(x_n) \to f(x)$.

6. Let (f_n) be a sequence of functions on a set $S \subset \mathbb{R}$ such that $f_n \to f$ uniformly on S. Prove that (f_n) is uniformly Cauchy on S.

7. Let (f_n) be a sequence of bounded functions on a set S such that $f_n \to f$ uniformly on S. Prove that f is bounded on S.

8. (a) Show that if $\sum |a_k| < \infty$, then $\sum a_k x^k$ converges uniformly on $[-1,1]$ to a continuous function.
   (b) Does $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ represent a continuous function on $[-1,1]$?

9. (a) Let $0 < a < 1$. Show that the series $\sum_{n=0}^{\infty} x^n$ converges uniformly on $[-a,a]$ to $\frac{1}{1-x}$.
   (b) Does the series $\sum_{n=0}^{\infty} x^n$ converge uniformly on $(-1,1)$ to $\frac{1}{1-x}$?

10. Let (f_n) be a sequence of continuous functions on $[a,b]$ such that $(f_n(x))$ is an increasing sequence of real numbers for each $x \in [a,b]$. Prove that if $f_n \to f$ pointwise on $[a,b]$ and if f is continuous on $[a,b]$, then $f_n \to f$ uniformly on $[a,b]$. (This is called *Dini's theorem.*)