Math 142B Homework Assignment 1
Due 11:00pm Wednesday, January 18, 2023

1. Let $f(x) = x^2 \sin\left(\dfrac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$.
   (a) Show that f is differentiable at each $x \neq 0$. (Use without proof the fact that $\sin(x)$ is differentiable and $\sin'(x) = \cos(x)$.)
   (b) Use the definition of derivative to show that f is differentiable at $x = 0$ and that $f'(0) = 0$.
   (c) Show that f' is not continuous at $x = 0$.

2. Let $f(x) = x^2 \sin\left(\dfrac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$, and $g(x) = x$ for $x \in \mathbb{R}$.
   (a) Calculate $f\left(\frac{1}{n\pi}\right)$ for $n = \pm1, \pm2, \pm3, \ldots$
   (b) Explain why $\displaystyle\lim_{x \to 0} \frac{g(f(x)) - g(f(0))}{f(x) - f(0)}$ is meaningless; that is, fails to exist.

3. Let $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
   (a) Show that f is continuous at $x = 0$.
   (b) Show that f is discontinuous at all $x \neq 0$.
   (c) Show that f is differentiable at $x = 0$. Note that the formula $f'(x) = 2x$ does *not* apply to this function f.

4. Suppose f is differentiable at $x = x_0$.
   (a) Show that $\displaystyle\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)$.
   (b) Show that $\displaystyle\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} = f'(x_0)$.

5. Suppose that the function $f : (0, \infty) \to \mathbb{R}$ is differentiable and let $c > 0$. Define $g : (0, \infty) \to \mathbb{R}$ by $g(x) = f(cx)$. Using only the definition of derivative (without appealing to the chain rule), show that $g'(x) = cf'(cx)$ for $x > 0$.

6. Let g be a function that is differentiable on an open interval I containing x_0.
   Define $h(x) = \begin{cases} \frac{g(x) - g(x_0)}{x - x_0} & \text{if } x \neq x_0, \\ g'(x_0) & \text{if } x = x_0. \end{cases}$
   (a) Show that h is continuous on I.
   (b) Show that if $g'(x_0) > 0$, then there is a $\delta > 0$ such that $\frac{g(x) - g(x_0)}{x - x_0} > 0$ for $0 < |x - x_0| < \delta$.

7. Show that $|\cos(x) - \cos(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

8. Let f be a function defined on $\mathbb{R}$ with the property that $|f(x) - f(y)| \leq (x - y)^2$. Show that f is a constant function.

9. Let f and g be differentiable functions defined on an open interval I. Suppose $a < b \in I$ with $f(a) = f(b) = 0$. Show that $f'(x) + f(x)g'(x) = 0$ for some $x \in (a, b)$. [Hint: Consider $h(x) = f(x)\, e^{g(x)}$.]

10. Show that $e\, x \leq e^x$ for all $x \in \mathbb{R}$.