A Note on Complex Differentials

1. Real Differentials

We have defined the differential of a differentiable real-valued function $f(x, y)$ by

$$df := \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Line integrals of a differential of a real-valued function are independent of path: $\int_\gamma df = f(B) - f(A)$ along any smooth path γ from the point A to the point B.

We say that the differential $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ is *exact.*

2. Complex Differentials

Differentials of complex-valued functions of a complex variable are related to differentials of real-valued functions of two real variables via the natural correspondence between $\mathbb{C}$ and $\mathbb{R}^2$ given by $z = x + iy \longleftrightarrow (x, y)$.

Theorem 1. *Given $h(z) = u(x, y) + iv(x, y)$ with continuous partial derivatives. Then,*

$$dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy.$$

The point: Under the natural correspondence between $\mathbb{C}$ and $\mathbb{R}^2$, the formula for the differential of a complex-valued function of a complex variable is identical in form to the formula for a real-valued function of two real variables.

Proof.

$$\begin{aligned} dh &= du + i\,dv \\ &= \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + i\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right) \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)dy \\ &= \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy. \end{aligned}$$

$\square$

Theorem 2. *If h is analytic, then* $dh = h'(z)\,dz$

Proof. Since h is analytic, we may write

$$h'(z) = \frac{\partial h}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{1}$$

Thus,

$$\begin{aligned}
h'(z)\,dz &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(dx + i\,dy) \\
&= \left(\frac{\partial u}{\partial x}dx - \frac{\partial v}{\partial x}dy\right) + i\left(\frac{\partial v}{\partial x}dx + \frac{\partial u}{\partial x}dy\right) \\
&= \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + i\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right) \quad \text{by the Cauchy-Riemann equations} \\
&= du + i\,dv \\
&= dh.
\end{aligned}$$

□

Exercise: When h is analytic, it is also true that

$$h'(z) = \frac{1}{i}\frac{\partial h}{\partial y}. \tag{2}$$

Starting with equation (2), verify that $dh = h'(z)\,dz$.