Math 120A
August 30, 2023

Question 1 Let $f(z) = \dfrac{1}{z^2+4} = \dfrac{1}{4} \cdot \dfrac{1}{1 + \left(\frac{z}{2}\right)^2}$.

A. $f(z) = \dfrac{1}{4} \sum_{k=0}^{\infty} (-1)^k \left(\dfrac{z}{2}\right)^{2k}$ is the power series for f centered at 0 and converges for $|z| < 2$.

B. $f(z)$ is analytic at ∞ since $g(w) = f\left(\frac{1}{w}\right) = \frac{w^2}{1+4w^2}$ is analytic at $w = 0$.

C. $f(z) = \dfrac{1}{4} \sum_{k=0}^{\infty} (-1)^k \left(\dfrac{2}{z}\right)^{2k+2}$ is the power series for f at ∞ and converges for $|z| > 2$.

D. **B** and **C**.

*E. All of the above.

Question 2 An analytic function $f(z)$ has a zero of order N at z_0 if

A. $f(z) = h(z)(z - z_0)^N$ for some analytic function $h(z)$ with $h(z_0) \neq 0$.

B. $f(z_0) = f'(z_0) = \cdots = f^{(N-1)}(z_0) = 0$ and $f^{(N)}(z_0) \neq 0$.

C. $f(z) = g(z)^N$ for some analytic function $g(z)$ with $g'(z_0) \neq 0$.

*D. All of the above.

E. None of the above. Zeros don't have any order.

Question 3 Given $R > 0$, let $\gamma = \left\{z \in \mathbb{C} \;\middle|\; |z| = R\right\}$. Then,

A. γ can be parametrized by $z(t) = Re^{it}$ with $0 \leq t < 2\pi$.

B. $\displaystyle\int_\gamma \left|z^2\right| \, dz = \int_0^{2\pi} R^2 \cdot R\, ie^{it}\, dt = R^3 e^{it}\Big|_{t=0}^{2\pi} = 0.$

C. $\displaystyle\int_\gamma \left|z^2\right| \, |dz| = \int_0^{2\pi} R^2 \cdot R\, dt = 2\pi R^3.$

*D. All of the above.

E. None of the above. Complex line integrals can't be real.

Question 4 To develop a power series $\sum_{k=0}^{\infty} a_k(z-i)^k$ centered at $z_0 = i$ for $f(z) = \frac{1}{z-1}$, one could write $f(z) = -\frac{1}{1-z} = -(1-z)^{-1}$ and:

A. Compute $f^{(k)}(z) = -k!(1-z)^{-(k+1)} = \frac{k!}{(1-z)^{k+1}}$ so that
$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(i)}{k!}(z-i)^k = \sum_{k=0}^{\infty} -\frac{1}{(1-i)^{k+1}}(z-i)^k.$$

B. Write $f(z) = -\frac{1}{(1-i)-(z-i)} = -\frac{1}{1-i} \cdot \frac{1}{1-\left(\frac{z-i}{1-i}\right)}$ so that
$$f(z) = -\frac{1}{1-i} \sum_{k=0}^{\infty} \left(\frac{z-i}{1-i}\right)^k = \sum_{k=0}^{\infty} -\frac{1}{(1-i)^{k+1}}(z-i)^k.$$

*C. Either **A** or **B**. Both are great ways to develop the power series.

D. Neither **A** nor **B**. $f(z) = \frac{1}{z-1}$ is not analytic on any disk centered at $z_0 = i$.

E. This selection has intentionally been left blank.

Question 5 The function $f(z) = \frac{1}{z} + \frac{1}{z^5}$ can be written $f(z) = \frac{z^4+1}{z^5}$. We can conclude that

A. $\dfrac{1}{z} + \dfrac{1}{z^5}$ is the Laurent series of f for $|z| > 0$.

B. $f(z)$ has four simple zeros: $z \in \left\{ e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}} \right\}$.

C. $f(z)$ has a simple zero at ∞.

D. $g(w) = f(1/w) = w(1 + w^4)$ has a simple zero at $w = 0$.

*E. All of the above.