Math 120A
August 29, 2023

Question 1 Suppose $\sum_{k=0}^{\infty} a_k z_0^k$ converges. We conclude that $\sum_{k=0}^{\infty} a_k z^k$

A. converges absolutely for every z with $|z| < |z_0|$.

B. converges uniformly for every z with $|z| \leq r$ whenever $r < |z_0|$.

C. converges absolutely for every z with $|z| = |z_0|$.

*D. **A** and **B**.

E. all of the above.

Question 2 Consider the power series $f(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^k$. Then,

A. $f(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges by alternating series test.

B. The radius of convergence $R \geq 1$ for $f(z)$ since it converges for $z = -1$ and $|-1| = 1$.

C. The radius of convergence $R = 1$ for $f(z)$ by the ratio test since $\lim_{k \to \infty} \frac{1/k}{1/(k+1)} = 1$.

D. $f(1) = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

*E. all of the above.

Question 3 Let $f(z) = \dfrac{1}{z^2+4} = \dfrac{1}{4} \cdot \dfrac{1}{1+\left(\frac{z}{2}\right)^2}$.

A. $f(z) = \dfrac{1}{4}\sum_{k=0}^{\infty}(-1)^k\left(\dfrac{z}{2}\right)^{2k}$ is the power series for f centered at 0 and converges for $|z| < 2$.

B. $f(z)$ is analytic at ∞ since $g(w) = f\left(\frac{1}{w}\right) = \frac{w^2}{1+4w^2}$ is analytic at $w = 0$.

C. $f(z) = \dfrac{1}{4}\sum_{k=0}^{\infty}(-1)^k\left(\dfrac{2}{z}\right)^{2k+2}$ is the power series for f at ∞ and converges for $|z| > 2$.

D. **B** and **C**.

*E. All of the above.

Question 4 Cauchy's integral formula states that a function $f(z)$ analytic on a bounded domain D and smooth on $D \cup \partial D$ satisfies

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} dw, \quad z \in D.$$

This is important because

A. the integral can be differentiated with respect to z.

B. by induction, one sees that for every $z \in D$,
$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{m+1}} dw$ for all integers $m \geq 0$.

C. one can conclude that a function $f(z)$ that has one complex derivative at each $z \in D$ must have complex derivatives of all orders at each $z \in D$.

*D. All of the above, and it's absolutely amazing.

E. None of the above, Cauchy's integral formula is as useless as it is aesthetically pleasing.

Question 5 It is a foundational result of complex analysis that a function $f(z)$ analytic on a disk $|z - z_0| < R$ is represented by a power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ with radius of convergence R. Why is this true?

A. That's what *analytic* means. It's the definition.

B. Real-valued functions with derivatives of all orders are analytic. It works the same for complex-valued functions.

C. So that the studying complex analysis can be made a little bit simpler.

*D. Cauchy's theorem allows us to write

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(w)}{(w - z_0)^{k+1}} dw \right) (z - z_0)^k$$

for each fixed $0 < r < R$.

E. None of the above. What's a power series?