Math 20E Homework Assignment 7
Due Tuesday, December 5, 2023

1. Let $\mathbf{r}(x, y, z) = (x,\ y,\ z)$ and $r(x, y, z) = \sqrt{x^2 + y^2 + z^2} = \|\mathbf{r}\|$. Verify the following identities.
   (a) $\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$.
   (b) $\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{r}}{r^3}\right) = 0$.
   (c) $\boldsymbol{\nabla} \times \mathbf{r} = \mathbf{0}$.

2. Let C be the closed, piecewise smooth curve formed by traveling in straight lines between the points $(0, 0, 0)$, $(2, 1, 5)$, $(1, 1, 3)$, and back to $(0, 0, 0)$ in that order. Use Stokes' theorem to evaluate the line integral
$$\int_C (xyz)\, dx + (xy)\, dy + (x)\, dz.$$

3. Evaluate the surface integral $\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S}$, where S is the portion the surface of a sphere defined by $x^2 + y^2 + z^2 = 1$ and $x + y + z \geq 1$, and where $\mathbf{F} = \mathbf{r} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$, with $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

4. Let $\mathbf{F} = x^2\,\mathbf{i} + (2xy + x)\,\mathbf{j} + z\,\mathbf{k}$. Let C be the circle $x^2 + y^2 = 1$ and S the disk $x^2 + y^2 \leq 1$ within the plane $z = 0$.
   (a) Determine the flux of $\mathbf{F}$ out of S.
   (b) Determine the circulation of $\mathbf{F}$ around C.
   (c) Find the flux of $\boldsymbol{\nabla} \times \mathbf{F}$. Verify Stokes' theorem directly in this case.

5. Let $\mathbf{F} = (0,\ -z,\ 1)$. Let S be the spherical cap $x^2 + y^2 + z^2 = 1$, where $z \geq \frac{1}{2}$.
   (a) Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ directly as a surface integral.
   (b) Verify that $\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{A}$, where $\mathbf{A} = (0,\ x,\ xz)$.
   (c) Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ using Stokes' theorem.

6. Let $\mathbf{F} = (y^2,\ x^2,\ z^2)$. Verify that $\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$, for any two simple closed curves $\mathcal{C}_1, \mathcal{C}_2$ going around a cylinder whose central axis is the z-axis; that is, any cylinder whose equation is of the form $x^2 + y^2 = R^2$.

7. Use the divergence theorem to calculate the flux of $\mathbf{F} = (x - y)\,\mathbf{i} + (y - z)\,\mathbf{j} + (z - x)\,\mathbf{k}$ out of the unit sphere $x^2 + y^2 + z^2 = 1$.

8. Let S be the boundary surface of a solid region W. Show that
$$\iint_S \mathbf{r} \cdot \mathbf{n}\, dS = 3 \text{ volume}(W).$$
   Explain this result geometrically.

9. Let W be the pyramid with top vertex $(0, 0, 1)$, and base vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 0)$. Let S be the closed boundary surface of W, oriented outward from W. Let $\mathbf{F}(x, y, z) = (x^2 y,\ 3y^2 z,\ 9z^2 x)$. Use Gauss' theorem to compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

10. Let $\mathbf{F}(x, y, z) = (x+y,\ z,\ z-x)$ and let $\mathcal{S}$ be the boundary surface of the solid region between the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.