Math 102 Homework Assignment 5
Due Thursday, February 17, 2022

1. Determine the QR factorization of $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

2. Let U be a m-dimensional subspace of $\mathbb{R}^n$ and let V be a k-dimensional subspace of U, where $0 < k < m$.
   (a) Show that any orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ for V can be extended to form an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_m\}$ for U.
   (b) Show that if $W = \text{Span}\,(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_m)$, then $U = V \oplus W$.

3. Let Q be an orthogonal matrix.
   (a) Show that if λ is an eigenvalue of Q, then $|\lambda| = 1$.
   (b) Show that $|\det(Q)| = 1$.

4. Let λ_1 and λ_2 be distinct eigenvalues of A. Let $\mathbf{x}$ be an eigenvector of A belonging to λ_1 and let $\mathbf{y}$ be an eigenvector of A^T belonging to λ_2. Show that $\mathbf{x} \perp \mathbf{y}$.

5. Let A and B be $n \times n$ matrices. Show that:
   (a) If λ is a nonzero eigenvalue of AB, then it is also an eigenvalue of BA.
   (b) If $\lambda = 0$ is an eigenvalue of AB, then $\lambda = 0$ is also an eigenvalue of BA.

6. Solve each of the following initial value problems:
   (a)
$$\begin{aligned} y_1' &= -y_1 + 2y_2 \qquad & y_1(0) &= 3 \\ y_2' &= 2y_1 - y_2 \qquad & y_2(0) &= 1 \end{aligned}$$
   (b)
$$\begin{aligned} y_1' &= y_1 - 2y_2 \qquad & y_1(0) &= 1 \\ y_2' &= 2y_1 + y_2 \qquad & y_2(0) &= -2 \end{aligned}$$

7. Given

$$\mathbf{Y} = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{x}_n$$

is the solution to the initial value problem

$$\mathbf{Y}' = A\mathbf{Y}, \qquad \mathbf{Y}(0) = \mathbf{Y}_0.$$

(a) Show that $\mathbf{Y}_0 = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \cdots + c_n\mathbf{x}_n$.

(b) Let $X = \begin{pmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$.

Given that the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent, show that $\mathbf{c} = X^{-1}\mathbf{Y}_0$.

8. (a) Transform the n^{th}-order equation $y^{(n)} = a_0\, y + a_1\, y' + \cdots + a_{n-1}\, y^{(n-1)}$ into a system of first-order equations by setting $y_1 = y$ and $y_{k+1} = y_k'$ $k = 1, \ldots, n-1$.

(b) Determine the characteristic polynomial of the coefficient matrix of this system.