

Math 120A  
Final Examination  
September 6, 2013  
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Instructions

1. You may use any type of calculator, but no other electronic devices during this exam.
  - Express numbers symbolically (for example,  $\sqrt{2}$  rather than 2.1).
2. You may use one page of notes, but no books or other assistance during this exam.
3. Write your *Name*, *PID*, and *Section* on the front of your Blue Book.
4. Write your solutions clearly in your Blue Book
  - (a) Carefully indicate the number and letter of each question and question part.
  - (b) Present your answers in the same order they appear in the exam.
  - (c) Start each question on a new side of a page.
5. Read each question carefully, and answer each question completely.
6. Show all of your work; no credit will be given for unsupported answers.

1. (6 points) Recall that  $\log(z) = \{w \mid e^w = z\}$  is *multivalued* and represents a *set* of values.

- (a) Verify that  $2 \log(i) \neq \log(i^2)$  as sets.
- (b) Verify that  $\frac{1}{2} \log(i) = \log(i^{1/2})$  as sets.

2. (6 points) Let  $u(x, y) = x^3 - 3xy^2$ .

- (a) Verify that  $u(x, y)$  is harmonic.
- (b) Find a function  $v(x, y)$  so that  $f(x + iy) = u(x, y) + iv(x, y)$  is analytic.

3. (6 points) The function

$$f(z) = \begin{cases} \frac{\sin(z)}{z} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0 \end{cases}$$

is entire.

- (a) Represent  $f(z)$  as a power series valid for all  $z$ . You may use the fact that the power series  $\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}$  is valid for all  $z$ .
- (b) Since  $f$  is entire,  $g(z) = f'(z) = \frac{z \cos(z) - \sin(z)}{z^2}$  for all  $z \neq 0$  has a removable singularity at  $z = 0$ . Determine the value of  $g(0)$  that makes  $g(z)$  an entire function.
- (c) Represent  $g(z)$  as a power series valid for all  $z$ .

**Note: Problems 4 – 7 are on the other side of this page.**

4. (6 points) Given  $f$  analytic on and within the circle  $C$  given by  $|z - z_0| = R$ . Use Cauchy's integral formula and the  $ML$ -inequality to show that if  $|f(z)| \leq M$  for all points  $z$  on  $C$ , then  $|f(z_0)| \leq M$ .

5. (6 points) Let  $f(z) = \frac{1}{z(z-1)}$ .

(a) Expand  $f(z)$  in a Laurent series valid in the annular domain  $0 < |z| < 1$ .

(b) Expand  $f(z)$  in a Laurent series valid in the domain  $|z| > 1$ .

6. (6 points) Evaluate the trigonometric integral

$$\int_0^{2\pi} \frac{1}{5 + 4 \sin(\theta)} d\theta$$

by representing it as a contour integral and using the residue theorem. Be sure to clearly specify the contour you use.

7. (6 points) Evaluate the Cauchy principle value

$$\int_{-\infty}^{\infty} \frac{2x^2 - 1}{x^4 + 5x^2 + 4} dx = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{2x^2 - 1}{x^4 + 5x^2 + 4} dx$$

as a limit of contour integrals, using the residue theorem. Be sure to clearly specify the contours you use and why the value of the integrals along the nonreal part of the contours tend to zero as  $R$  tends to infinity.