A Note on Differentiability

1. Real-Differentiability

Recall that in one-variable calculus, we define differentiability as follows:

Definition 1. *A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0 if $\displaystyle\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = m$ for some number $m \in \mathbb{R}$. When this occurs, we define $f'(x_0) := m$ and call $f'(x_0)$ the derivative of f at x_0.*

Unfortunately, this definition does not generalize to functions $f : \mathbb{R}^n \to \mathbb{R}^m$. However, the following equivalent definition does generalize to functions $f : \mathbb{R}^n \to \mathbb{R}^m$.

Definition 2. *A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0 if*

$$\lim_{|\Delta x| \to 0} \frac{|f(x_0 + \Delta x) - f(x_0) - m\,\Delta x|}{|\Delta x|} = 0 \quad \textit{for some number } m \in \mathbb{R}.$$

When this occurs, we define $f'(x_0) := m$ and call $f'(x_0)$ the derivative of f at x_0.

The reason the above two definitions are equivalent can be seen by the following computation:

$$\begin{aligned}
\lim_{|\Delta x| \to 0} \frac{|f(x_0 + \Delta x) - f(x_0) - m\,\Delta x|}{|\Delta x|} &= \lim_{|\Delta x| \to 0} \left| \frac{f(x_0 + \Delta x) - f(x_0) - m\,\Delta x}{\Delta x} \right| \\
&= \lim_{|\Delta x| \to 0} \left| \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - m \right|.
\end{aligned}$$

The second definition generalizes naturally to functions of the form $f : \mathbb{R}^2 \to \mathbb{R}^2$.

Definition 3. *A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (u(x, y),\, v(x, y))$ is differentiable at (x_0, y_0) if there is a 2×2 matrix M such that*

$$\lim_{|(\Delta x, \Delta y)| \to 0} \frac{\left| \Big(u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0),\ v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0) \Big) - M \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \right|}{|(\Delta x, \Delta y)|} = 0.$$

When this occurs, we define $Df(x_0, y_0) := M$ and call $Df(x_0, y_0)$ the derivative of f at (x_0, y_0). When the point (x_0, y_0) is clear from the context, we often simply write Df for the derivative of f.

We used the following notational conventions in the above definition.

1. $|(\Delta x, \Delta y)| := \sqrt{\Delta x^2 + \Delta y^2}$.
2. $\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = (\Delta x, \Delta y)$
   - ordered pairs may be written as either columns or rows.
3. $M \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} m_{11}\Delta x + m_{12}\Delta y \\ m_{21}\Delta x + m_{22}\Delta y \end{pmatrix} = (m_{11}\Delta x + m_{12}\Delta y,\ m_{21}\Delta x + m_{22}\Delta y)$.
   - ordered pairs are written as columns in order to perform matrix multiplication.

Theorem 1. *If $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = \big(u(x, y),\, v(x, y)\big)$ has continuous partial derivatives, then f is differentiable, and the derivative Df is the matrix of partial derivatives, also called J_f, the Jacobian matrix of f.*

$$Df = J_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

When $Df(x_0, y_0) = J_f(x_0, y_0)$ is the derivative of f at (x_0, y_0), the partial derivatives are all evaluated at (x_0, y_0).

Complex-Differentiability

Every function $f : \mathbb{C} \to \mathbb{C}$ given by $f(x + i\,y) = u + i\,v$ corresponds to a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (u(x, y),\, v(x, y))$.

If $f : \mathbb{C} \to \mathbb{C}$ is analytic, then

$$\lim_{\Delta z \to 0} \frac{f(z + z_0) - f(z_0) - f'(z_0)\Delta z}{\Delta z} = 0. \tag{1}$$

By comparing equation (1) and Definition 3, we see that $Df(x_0, y_0) \cdot \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = f'(z_0) \cdot \Delta z$ under the natural correspondence $\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = (\Delta x, \Delta y) = \Delta z$ between $\mathbb{R}^2$ and $\mathbb{C}$.

The obvious question is:

Question: What is the difference between $f'(z_0)$ and $Df(x_0, y_0)$?

1. Comparing Definition 3 and Equation 1, we see that $(\Delta x,\ \Delta y) \leftrightarrow \Delta z$ and $\mathrm{Df}(x_0, y_0) \leftrightarrow f'(z_0)$. If we write $f'(z_0) = \delta_1 + i\delta_2$ and $\Delta z = \Delta x + i\Delta y$, we see that

$$\begin{aligned} f'(z_0)\Delta z &= (\delta_1 + i\delta_2)(\Delta x + i\Delta y) \\ &= (\delta_1 \Delta x - \delta_2 \Delta y) + i(\delta_2 \Delta x + \delta_1 \Delta y) \\ &= \begin{pmatrix} \delta_1 \Delta x - \delta_2 \Delta y \\ \delta_2 \Delta x + \delta_1 \Delta y \end{pmatrix} \\ &= \begin{pmatrix} \delta_1 & -\delta_2 \\ \delta_2 & \delta_1 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \mathrm{Df}(x_0, y_0) \cdot \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}. \end{aligned}$$

We conclude that when f is complex-differentiable at $z_0 = (x_0, y_0)$ (that is, $f'(z_0)$ exists), then $\mathrm{Df} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \delta_1 & -\delta_2 \\ \delta_2 & \delta_1 \end{pmatrix}$ and the partial derivatives satisfy

$$\frac{\partial u}{\partial x} = \delta_1 = \frac{\partial v}{\partial y}; \text{ and } \frac{\partial u}{\partial y} = \delta_2 = -\frac{\partial v}{\partial x}. \text{ (the Cauchy-Riemann equations)}$$

2. Your book records this as

Theorem 1. *If $f(z)$ is analytic, then its Jacobian matrix $J_f = Df$ (as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$) has determinant*

$$\det J_f(z) = |f'(z)|^2.$$

3. A (real) differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ need not be complex-differentiable. A simple example of this is the function $f(x, y) = (x, -y)$; that is, $f(z) = \bar{z}$. Then, $\mathrm{Df} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, but the partial derivatives of f do not satisfy the Cauchy-Riemann equations. So, $f(z) = \bar{z}$ is real-differentiable but not complex-differentiable.

The point of all this is that the algebraic structure placed on the complex plane introduces an additional constraint on complex-differentiability that is not required for real-differentiability. As we shall see, this additional constraint leads to far-reaching consequences.