# A Note on Complex Differentials

## 1. Real Differentials

We have defined the differential of a differentiable real-valued function $f(x, y)$ by

$$df := \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Line integrals of a differential of a real-valued function are independent of path: $\int_\gamma df = f(B) - f(A)$ along any smooth path $\gamma$ from the point $A$ to the point $B$.

We say that the differential $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ is *exact*.

## 2. Complex Differentials

Differentials of complex-valued functions of a complex variable are naturally related to differentials of real-valued functions of two real variables via the natural correspondence between $\mathbb{C}$ and $\mathbb{R}^2$ given by $z = x + iy \longleftrightarrow (x, y)$.

**Theorem 1.** *Given* $h(z) = u(x, y) + iv(x, y)$ *with continuous partial derivatives. Then,*

$$dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy.$$

The point: Under the natural correspondence between $\mathbb{C}$ and $\mathbb{R}^2$, the formula for the differential of a complex-valued function of a complex variable is identical in form to the formula for a real-valued function of two real variables.

*Proof.*

$$\begin{aligned} dh &= du + i\, dv \\ &= \left( \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) + i \left( \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right) \\ &= \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) dx + \left( \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) dy \\ &= \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy. \end{aligned}$$

□

**Theorem 2.** *If $h$ is analytic, then* $dh = h'(z)\, dz$

*Proof.* Since $h$ is analytic, we may write

$$h'(z) = \frac{\partial h}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Thus,

$$\begin{aligned}
h'(z)\,dz &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(dx + i\,dy) \\
&= \left(\frac{\partial u}{\partial x}dx - \frac{\partial v}{\partial x}dy\right) + i\left(\frac{\partial v}{\partial x}dx + \frac{\partial u}{\partial x}dy\right) \\
&= \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + i\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right) \quad \text{by the Cauchy-Riemann equations} \\
&= du + i\,dv \\
&= dh.
\end{aligned}$$

$\square$