Math 109: A List of Supplementary Exercises

1. Let b be a nonzero integer and let a, q and r be integers such that $a = bq + r$. Prove that $\gcd(a, b) = \gcd(b, r)$.

2. Let n be a positive integer and let a be an integer coprime to n. Prove that for every integer b, there is an integer x such that $ax - b$ is divisible by n.

3. Let a, b and c be integers such that $\gcd(a, c) = \gcd(b, c) = 1$. Prove that $\gcd(ab, c) = 1$.

4. Let a, b and c be integers such that a and b are coprime and c divides $a + b$. Prove that $\gcd(a, c) = \gcd(b, c) = 1$.

5. Show that $\gcd(5n + 2, 12n + 5) = 1$ for every integer n.

6. Let p and q be integers such that 3 divides $p^2 + q^2$. Prove that 3 divides p and 3 divides q.

7. Find a positive integer n and members $[a]$ and $[b]$ of $\mathbb{Z}_n$ such that $[a] \cdot [b] = [0]$ but $[a] \neq [0]$ and $[b] \neq [0]$.

8. Prove that the nonzero element $[a]$ of $\mathbb{Z}_n$ has a multiplicative inverse in $\mathbb{Z}_n$ if and only if n and a are coprime.

9. Define $\simeq$ on $\mathbb{R}$ by $x \simeq y$ if and only if $x - y \in \mathbb{Z}$.

   (a) Prove that $\simeq$ is an equivalence relation on $\mathbb{R}$.

   (b) Which real numbers belong to $[-17]$?

   (c) Characterize the partition Π on $\mathbb{R}$ corresponding to $\simeq$.

10. Define $\sim$ on the set $M_{n\times n}$ of all $n \times n$ matrices by $A \sim B$ if and only if there is an invertible matrix $P \in M_{n\times n}$ such that $B = P^{-1}AP$. Prove that $\sim$ is an equivalence relation on $M_{n\times n}$.

11. For each real number b, let $A_b = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = |x + b|\}$, and let $\mathcal{A} = \{A_b \mid b \in \mathbb{R}\}$. Is $\mathcal{A}$ a partition of $\mathbb{R} \times \mathbb{R}$? Justify your answer.

12. For each real number b, let $A_b = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = |x| + b\}$, and let $\mathcal{A} = \{A_b \mid b \in \mathbb{R}\}$. Is $\mathcal{A}$ a partition of $\mathbb{R} \times \mathbb{R}$? Justify your answer.

13. Let $f : A \to A$ be a function from a set A to itself.

    (a) Given that A is finite, prove that f is injective if and only if f is surjective.

    (b) Let $A = \mathbb{Z}^+$. Find a function $f_1 : A \to A$ that is injective but not surjective, and find a function $f_2 : A \to A$ that is surjective but not injective.

14. Since $(0, 1)$ and $[0, 1]$ have the same cardinality, there must be a bijection $\sigma : (0, 1) \to [0, 1]$ between them. Find an explicit formula for one.

15. Let $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$. ($\mathbb{N}$ is often called the set of natural numbers.) Let $\mathcal{F}(\mathbb{N})$ be the collection of all *finite* subsets of $\mathbb{N}$. Find an explicit bijection $\sigma : \mathcal{F}(\mathbb{N}) \to \mathbb{N}$. (Hint: Think about binary representation of integers.)