Math 20E Homework Assignment 5
Due Monday, November 14, 2022

1. A metallic surface S is in the shape of a hemisphere $z = \sqrt{R^2 - x^2 - y^2}$, where (x, y) satisfies $x^2 + y^2 \leq R^2$. The mass density (mass per unit area) at $(x, y, z) \in S$ is given by $m(x, y, z) = x^2 + y^2$. Find the total mass of S.

2. Find the average value of $f(x, y, z) = x + z^2$ on the truncated cone $z^2 = x^2 + y^2$, with $3 \leq z \leq 4$.

3. Evaluate the integral $\iint_S (1-z)\, dS$, where S is the graph of $z = 1-x^2-y^2$, with $x^2+y^2 \leq 1$.

4. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, with $\mathbf{F}(x, y, z) = (x, y, z)$, and S the part of the plane $x + y + z = 1$ with $x \geq 0$, $y \geq 0$, and $z \geq 0$.

5. Let $\mathcal{S}$ be the ellipsoid $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$. Compute the flux of $\mathbf{F} = (0, 0, z)$ over the portion of $\mathcal{S}$ where $x \leq 0,\ y \leq 0,\ z \leq 0$ with upward-pointing normal.

6. Let $\mathbf{v} = (0, 0, z)$ be the velocity field (in meters per second) in $\mathbb{R}^3$. Compute the volume flow rate (in cubic meters per second) through the upper upper hemisphere ($z \geq 0$) of the unit sphere $x^2 + y^2 + z^2 = 1$.

7. A net with surface described by $y = 0$ with $x^2 + z^2 \leq 1$ is dipped into a river in which the water flows according to the velocity field $\mathbf{v} = (x - y,\ z + y + 4,\ z^2)$. Determine the volume flow rate across the net.

8. Compute the area enclosed by the ellipse $\left(\frac{x}{c}\right)^2 + \left(\frac{y}{d}\right)^2 = 1$.

9. Find the area of the region between the x-axis and the cycloid parametrized by $\mathbf{r}(t) = (t - \sin(t),\ 1 - \cos(t))$ with $0 \leq t \leq 2\pi$.

10. Let $P(x, y) = \dfrac{-y}{x^2 + y^2}$ and $Q(x, y) = \dfrac{x}{x^2 + y^2}$, and let D be the unit disk $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

    (a) Evaluate the area integral $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$ over the unit disk D.

    (b) Evaluate the line integral $\int_{\partial D} P\, dx + Q\, dy$ around ∂D, the unit circle with positive orientation.

    (c) Briefly explain why Green's theorem failed.