Math 142A Homework Assignment 6
Due Friday, December 2, 2022

1. Let k be a natural number. Prove that $\lim\limits_{x\to 1} \dfrac{x^k - 1}{x - 1} = k$.

2. Suppose that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is periodic; that is, there is a number $p > 0$ such that $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

3. A function $f : D \to \mathbb{R}$ is said to be *Lipschitz* provided that there is a number $C \geq 0$ with

$$|f(u) - f(v)| \leq C\,|u - v| \quad \text{for all } u \text{ and } v \text{ in } D.$$

Show that if a function f is a Lipschitz function on D, then it is *uniformly* continuous on D.

4. Prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S.

5. Let f be a continuous function on $[0, \infty)$. Prove that if f is uniformly continuous on $[k, \infty)$ for some $k > 0$, then f is uniformly continuous on $[0, \infty)$.

6. Let f be a continuous function on $[a, b]$. Show that the function f^* defined by

$$f^*(x) = \sup\{f(y) \mid a \leq y \leq x\}, \text{ for } x \in [a, b],$$

is an increasing continuous function on $[a, b]$.

7. Suppose the limits $L_1 = \lim\limits_{x\to a^+} f_1(x)$ and $L_2 = \lim\limits_{x\to a^+} f_2(x)$ both exist.
   (a) Show that if $f_1(x) \leq f_2(x)$ for all x in some interval (a, b), then $L_1 \leq L_2$.
   (b) Suppose $f_1(x) < f_2(x)$ for all x in some interval (a, b). Can you conclude that $L_1 < L_2$?

8. Show that if $\lim\limits_{x\to a^+} f_1(x) = \lim\limits_{x\to a^+} f_3(x) = L$ and if $f_1(x) \leq f_2(x) \leq f_3(x)$ for all x in some interval (a, b), then $\lim\limits_{x\to a^+} f_2(x) = L$. (Note: Be sure to show that $\lim\limits_{x\to a^+} f_2(x)$ exists, since this is not assumed.)

9. Let $f(x) = \dfrac{\sqrt{1+3x^2} - 1}{x^2}$ for $x \neq 0$. Show that $\lim\limits_{x\to 0} f(x)$ exists and determine its value. Be sure to justify all claims.

10. Let f_1 and f_2 be functions such that $\lim\limits_{x\to a^S} f_1(x) = +\infty$ and such that the limit $L_2 = \lim\limits_{x\to a^S} f_2(x)$ exists.
    (a) Prove that $\lim\limits_{x\to a^S} (f_1 + f_2)(x) = +\infty$ if $L_2 \neq -\infty$.
    (b) Prove that $\lim\limits_{x\to a^S} (f_1\, f_2)(x) = +\infty$ if $0 < L_2 \leq +\infty$.
    (c) Prove that $\lim\limits_{x\to a^S} (f_1\, f_2)(x) = -\infty$ if $-\infty \leq L_2 < 0$.
    (d) What can you say about $\lim_{x\to a^S} (f_1\, f_2)(x)$ if $L_2 = 0$?