Math 142A Homework Assignment 5
Due Friday, November 18, 2022

1. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous at the point x_0 and that $f(x_0) > 0$. Show that there is an interval $I_n = \left(x_0 - \frac{1}{n},\ x_0 + \frac{1}{n}\right)$ for some $n \in \mathbb{N}$ for which $f(x) > 0$ for every $x \in I_n$.

2. A function $f : D \to \mathbb{R}$ is said to be *Lipschitz* provided that there is a number $C \geq 0$ with

$$|f(u) - f(v)| \leq C\,|u - v| \quad \text{for all } u \text{ and } v \text{ in } D.$$

Show that a Lipschitz function is continuous.

3. Suppose the function $\lambda : \mathbb{R} \to \mathbb{R}$ has the the property that $\lambda(u+v) = \lambda(u) + \lambda(v)$ for all u, v.
   (a) Define the number m by $m := \lambda(1)$. Show that $\lambda(x) = m\,x$ for all rational numbers x.
   (b) Show that if λ is continuous, then $\lambda(x) = m\,x$ for all $x \in \mathbb{R}$.

4. (a) Let $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$
   Show that f is discontinuous at every $x \in \mathbb{R}$.

   (b) Let $h(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$
   Show that h is continuous at $x = 0$ and at no other point.

5. Let f be a real-valued function whose domain is a subset of $\mathbb{R}$.
   Show that f is continuous at $x_0 \in \mathrm{dom}(f)$ if and only if for every sequence (x_n) in $\mathrm{dom}(f) \setminus \{x_0\}$ converging to x_0, we have $\lim\limits_{n \to \infty} f(x_n) = f(x_0)$.

6. Let f and g be continuous functions on $[a,\ b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Show that there is at least one $x_0 \in [a,\ b]$ at which $f(x_0) = g(x_0)$.

7. Prove that a polynomial $p(x)$ with odd degree has a least one real zero.

8. Suppose that the function $f : [a,\ b] \to \mathbb{R}$ is continuous. Show that for any $n \in \mathbb{N}$ and points $x_1, x_n, \ldots, x_n \in [a,\ b]$, there is a point $z \in [a,\ b]$ such that $f(z) = \dfrac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$.

9. Suppose that the function $f : [0,\ 1] \to \mathbb{R}$ is continuous with $f(0) > 0$, and $f(1) = 0$. Show that there is a number $x_0 \in (0,\ 1]$ such that $f(x_0) = 0$ and $f(x) > 0$ for every $x \in [0,\ x_0)$.

10. Let $f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sin\left(\frac{1}{x}\right) & \text{otherwise.} \end{cases}$
    Show that f has the intermediate value property on all of $\mathbb{R}$.