**Math 142A Homework Assignment 4**
**Due Friday, November 4, 2022**

1. Let $(s_n)$ and $(t_n)$ be bounded sequences of nonnegative real numbers. Prove that

$$\limsup s_n t_n \le (\limsup s_n)(\limsup t_n).$$

2. Prove that $(s_n)$ is bounded if and only if $\limsup |s_n| \in \mathbb{R}$ (that is, $\limsup |s_n| < +\infty$).

3. Let $(s_n)$ be a bounded sequence of nonzero real numbers. Prove that

$$\liminf \left| \frac{s_{n+1}}{s_n} \right| \le \liminf |s_n|^{1/n}.$$

4. Let $(s_n)$ be a sequence of nonnegative numbers. For each $n$, define $\sigma_n = \dfrac{1}{n}(s_1 + s_2 + \cdots + s_n)$. [Recall from Homework 2 that $(\sigma_n)$ is called the sequence of Cesàro means for $(s_n)$.]
   (a) Show that $\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n$.
   (b) Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.
   (c) Exhibit an example for which $\lim \sigma_n$ exists, but $\lim s_n$ does not exist.

5. Show that if $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers, then $\sum \sqrt{a_n b_n}$ converges.

6. Find a series $\sum a_n$ which diverges by the Root Test but for which the Ratio Test gives no information.

7. Let $(a_n)$ be a sequence of nonzero real numbers such that the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ is a constant sequence. Show that $\sum a_n$ is a geometric series.

8. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $\liminf |a_n| = 0$. Prove there is a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $\displaystyle\sum_{k=1}^{\infty} a_{n_k}$ converges.

9. (a) Exhibit an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.
   (b) Show that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ also converges.
   (c) Exhibit an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

10. Prove that if $(a_n)$ is a decreasing sequence of positive real numbers and if $\sum a_n$ converges, then $\displaystyle\lim_{n\to\infty} n a_n = 0$