Math 142A Homework Assignment 2
Due Friday, October 14, 2022

1. Suppose that $s_n \neq 0$ for every index n and that the limit $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ is defined.
   (a) Show that if $L < 1$, then $\lim s_n = 0$.
   (b) Show that if $L > 1$, then $\lim |s_n| = +\infty$.
   (See Exercise 9.12 in your text for a hint.)

2. Let $s_1 = 1$, and for $n \geq 1$, let $s_{n+1} = \sqrt{s_n + 1}$. It turns out that (s_n) converges. Assume this fact and show that limit $\lim s_n = \frac{1}{2}\left(1 + \sqrt{5}\right)$.

3. Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$.
   (a) Show that if $a = \lim x_n$, then $a = \frac{1}{3}$ or $a = 0$.
   (b) Does $\lim x_n$ exist? Justify your answer.
   (c) Explain the apparent contradiction between the result in part (a) and part (b).

4. Show that $\displaystyle\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

5. (a) Verify that $1 + a + a^2 + \cdots + a^n = \dfrac{1 - a^{n+1}}{1 - a}$ for $a \neq 1$.
   (b) Determine $\displaystyle\lim_{n\to\infty} \left(1 + a + a^2 + \cdots + a^n\right)$ for $|a| < 1$.
   (c) What is $\displaystyle\lim_{n\to\infty} \left(1 + a + a^2 + \cdots + a^n\right)$ for $a \geq 1$?

6. Let S be a bounded nonempty subset of $\mathbb{R}$ such that $\sup(S) \notin S$. Show that there is a sequence (s_n) of points in S such that $\lim s_n = \sup(S)$.

7. Let (s_n) be a sequence such that $|s_{n+1} - s_n| < 2^{-n}$ for all $n \in \mathbb{N}$.
   (a) Prove that (s_n) is a Cauchy sequence and, therefore, a convergent sequence.
   (b) Is (s_n) a Cauchy sequence if we only assume that $|s_{n+1} - s_n| < \dfrac{1}{n}$ for all $n \in \mathbb{N}$?

8. Let (s_n) be a sequence. Define the sequence (σ_n) by $\sigma_n = \dfrac{s_1 + s_2 + \cdots + s_n}{n}$.
   (σ_n) is called the sequence of Cesàro means for (s_n).
   (a) Show that if (s_n) is an increasing sequence, then (σ_n) is an increasing sequence.
   (b) Show that if $s_n \to s$, then $\sigma_n \to s$.

9. Your textbook's proof of Theorem 10.2 only proves that bounded increasing sequences converge.
   Prove: All bounded *decreasing* sequences converge.

10. (a) Let (s_n) be a monotone sequence. Prove that (s_n) converges if and only if (s_n^2) converges.
    (b) Find a non-monotone sequence (t_n) such that (t_n^2) converges but (t_n) does not converge.