Math 142A Homework Assignment 4
Due Thursday, February 11, 2021

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at x_0 with $f(x_0) > 0$. Prove that there is a natural number n for which $f(x) > 0$ for all x in the interval $I := (x_0 - 1/n, x_0 + 1/n)$.

2. Let S be a nonempty set of real numbers that is *not* sequentially compact. Prove that there is an unbounded sequence in S or there is a sequence in S that converges to a point x_0 which is not in S.

3. Let $f : [0,1] \to \mathbb{R}$ be continuous with $f(0) > 0$ and $f(1) = 0$. Prove that there is an x_0 in $(0,1]$ such that $f(x_0) = 0$ and $f(x) > 0$ for all x in $[0, x_0)$; that is, there is a smallest point in the interval $[0,1]$ at which f attains the value 0.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function whose image $f(\mathbb{R})$ is bounded. Prove that there is a solution to the equation $f(x) = x$.

5. Let $f : [a,b] \to \mathbb{R}$ be continuous. Given a natural number k, let $x_1, \ldots, x_k$ be points in $[a,b]$. Prove that there is a point z in $[a,b]$ at which

$$f(z) = \frac{f(x_1) + \cdots + f(x_k)}{k}.$$

   [Note: As $k \to \infty$, this becomes the mean value theorem for integrals (Theorem 6.26).]

6. A function $f : D \to \mathbb{R}$ is called a *Lipschitz function* if there is a $C \geq 0$ such that

$$|f(u) - f(v)| \leq C\,|u - v| \quad \text{for all } u, v \in D.$$

   Prove that if f is a Lipschitz function, then f is uniformly continuous.