Math 142A Homework Assignment 3
Due Thursday, January 28, 2021

1. Let $\{a_n\}$ be a monotone sequence.
   (a) Prove that $\{a_n\}$ converges if and only if $\{a_n^2\}$ converges.
   (b) Show that the result in part (a) is false if $\{a_n\}$ is not monotone.

2. Consider the sequence $\{a_n\}$ defined as follows:

$$\begin{cases} a_1 & = 2, \\ a_{n+1} & = \frac{2a_n+1}{a_n+1} \text{ for } n \geq 1. \end{cases}$$

   (a) Prove that $\{a_n\}$ is monotonically decreasing.
   (b) Prove that $\{a_n\}$ is bounded.
   (c) Explain how you know that $\{a_n\}$ converges and determine $\lim_{n\to\infty} a_n$.

3. Let $h_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$ be the n^{th} harmonic sum.
   (a) Show by induction that $h_{2^n} \geq 1 + \dfrac{n}{2}$ for every index n.
   (b) Show that the sequence $\{h_n\}$ is unbounded.

4. Show that a strictly increasing sequence has no peak indices.

5. Show that every index of a monotonically decreasing sequence is a peak index.

6. Suppose $\{a_n\}$ is a monotone sequence with a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to a$.
   (a) Prove that $a_n \to a$.
   (b) Exhibit an example showing that the result of part (a) is false if $\{a_n\}$ is not monotone.