Math 142A Homework Assignment 2
Due Thursday, January 21, 2021

1. Given a sequence $\{a_n\}$. Prove that $\lim_{n\to\infty} |a_n| = \infty$ if and only if $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

2. Let $\{a_n\}$ and $\{b_n\}$ be sequences. Prove the following statements.
   (a) If $a_n \to L$ and $a_n \le K$ for every index n, then $L \le K$.
   (b) If $a_n \to L$, $b_n \to K$, and $a_n \le b_n$ for every index n, then $L \le K$.
   (c) If $a_n \to L$, $b_n \to L$, and $a_n \le c_n \le b_n$ for every n, then $c_n \to L$.

3. Given a sequence $\{c_n\}$. Prove that $c_n \to c$ if and only if $c_n - c \to 0$.

4. Prove that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

   Hint: Set $\alpha_n = n^{\frac{1}{n}} - 1$ and show that $n = (1+\alpha_n)^n \ge 1 + \frac{n(n-1)}{2}\alpha_n^2$ for every index n by applying the Binomial Formula.

5. Show that the set $(-\infty, 0]$ is closed.

6. (a) Show that every real number is the limit of a sequence of irrational numbers.
   (b) Show that the set of irrational numbers fails to be closed.