1. Suppose that z_0 is an isolated singularity of $f(z)$ and that $(z-z_0)^N f(z)$ is bounded near z_0. Show that z_0 is either removable or a pole of order at most N.

2. Suppose that z_0 is an isolated singularity of $f(z)$ that is not removable. Show that z_0 is an essential singularity of $e^{f(z)}$.

3. Show that if $f(z)$ is a nonconstant entire function, then $e^{f(z)}$ has an essential singularity at $z=\infty$.

4. Obtain the partial fraction decomposition of $f(z)=\dfrac{z^6}{(z^2+1)(z-1)^2}$.

5. $g(z)=e^{\frac{1}{z}}$ has an essential singularity at 0.
   (a) Compute the residue of $g(z)$ at 0.
   (b) Evaluate the integral $\displaystyle\oint_{|z|=2} g(z)\,dz$.

6. Evaluate $\displaystyle\oint_{|z|=2} \frac{z}{\cos(z)}\,dz$.

7. Suppose $P(z)$ and $Q(z)$ are polynomials with $\deg(P)<\deg(Q)$ and such that the zeros of $Q(z)$ are simple zeros at the points $z_1,\ldots,z_N$. Show that the partial fraction decomposition of $f(z)=\dfrac{P(z)}{Q(z)}$ is given by

$$f(z)=\sum_{k=1}^{N}\frac{P(z_k)}{Q'(z_k)}\cdot\frac{1}{z-z_k}.$$

8. Consider the integral $\displaystyle\int_{\partial D_R}\frac{e^{\pi i(z-1/2)^2}}{1-e^{-2\pi i z}}\,dz$, where D_R is the parallelogram with vertices $-\left(\frac{1}{2}\right)-(1+i)R,\quad \left(\frac{1}{2}\right)-(1+i)R,\quad \left(\frac{1}{2}\right)+(1+i)R,\quad -\left(\frac{1}{2}\right)+(1+i)R.$
   (a) Use the residue theorem to show that the integral is equal to $\dfrac{1+i}{\sqrt{2}}$.
   (b) By parametrizing the sides of the parallelogram, show that the integral tends to $(1+i)\displaystyle\int_{-\infty}^{\infty} e^{-2\pi t^2}\,dt$ as $R\to\infty$.
   (c) Use parts (a) and (b) above to show that $\displaystyle\int_{-\infty}^{\infty} e^{-s^2}\,ds=\sqrt{\pi}$.
   [Note: if it's easier for you, you may show that $\displaystyle\int_{-\infty}^{\infty} e^{-2\pi t^2}\,dt=\frac{1}{\sqrt{2}}$].

9. Show that $\displaystyle\int_{-\infty}^{\infty}\frac{x}{(x^2+2x+2)(x^2+4)}\,dx=-\frac{\pi}{10}$.

10. Given $a>0$, show that $\displaystyle\int_{-\infty}^{\infty}\frac{\cos(ax)}{x^4+1}\,dx=\frac{\pi}{\sqrt{2}}e^{-\frac{a}{\sqrt{2}}}\left[\cos\left(\frac{a}{\sqrt{2}}\right)+\sin\left(\frac{a}{\sqrt{2}}\right)\right]$.

11. Show that $\displaystyle\int_{0}^{2\pi}\frac{1}{a+b\sin(\theta)}\,d\theta=\frac{2\pi}{\sqrt{a^2-b^2}}$, for $0<b<a$.

12. Show that $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\dfrac{1-r^2}{1-2r\cos(\theta)+r^2}\,d\theta = 1,$ for $0 \le r < 1$.

[*Remark*: The integrand $\frac{1-r^2}{1-2r\cos(\theta)+r^2}$ is called the Poisson kernel.]