1. Find the radius of convergence of each of the following series:
   (a) $\displaystyle\sum_{n=0}^{\infty} z^{3^n} = z + z^3 + z^9 + z^{27} + z^{81} + z^{243}\cdots$
   (b) $\displaystyle\sum_{p \text{ prime}} z^p = z^2 + z^3 + z^5 + z^7 + z^{11} + z^{13} + \cdots$

2. Define $f(z) = \displaystyle\sum_{n=0}^{\infty} z^{n!}$.
   (a) Show that $f(z)$ is analytic on the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$.
   (b) Show that $|f(r\lambda)| \to +\infty$ as $r \to 1$ whenever λ is a root of unity.

3. Define $f(z) := \text{Log}(z)$.
   (a) Find the power series expansion of $f(z)$ about the point $z = i - 2$ and show that the radius of convergence is $R = \sqrt{5}$.
   (b) Explain why this does not contradict the fact that $\text{Log}(z)$ has a discontinuity at $z = -2$.

4. Find the power series expansion of the principal branch $\text{Tan}^{-1}(z)$ of the inverse tangent function about $z = 0$. What is the radius of convergence?

   Hint: Integrate the power series expansion of its derivative term by term.

5. Let E be a bounded subset of $\mathbb{C}$ over which area integrals can be defined, and set

$$f(w) := \iint_E \frac{1}{w - z}\, dx\, dy; \quad \text{where } z = x + i\,y \text{ and } w \in \mathbb{C} \setminus E.$$

   Show that $f(w)$ is analytic at ∞, and find a formula for the coefficients of the power series of (w) at ∞ in descending powers of w.

   [Hint: Try a geometric series expansion.]

6. Calculate the terms up to order five of the power series expansion about $z = 0$ of the function $f(z) = \dfrac{z}{\sin(z)}$.

7. Define $f(z) = \dfrac{e^z}{1+z}$. Show that:
   (a) $f(z) = 1 + \frac{1}{2}z^2 - \frac{1}{3}z^3 + \frac{3}{8}z^4 - \frac{11}{30}z^5 + \cdots$.
   (b) The general n^{th} term of the power series is given by $a_n = (-1)^n \left[\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right]$, for $n \geq 2$.
   (c) What is the radius of convergence of the series?

8. Show that the zeros of $\sin(z)$ are all simple.

9. Show that $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$, assuming only that the identity holds for z and w real.

10. Show that if the analytic function $f(z)$ has a zero of order N at z_0, then $f(z) = g(z)^N$ for some function $g(z)$ analytic near z_0 and satisfying $g'(z_0) \neq 0$.

11. For each of the following functions, find the Laurent expansion centered at $z = -1$ that converges at $z = \frac{1}{2}$, and determine the largest open set (disk) on which the series converges.

    (a) $\dfrac{1}{z^2 - z}$

    (b) $\dfrac{z-1}{z+1}$

12. Suppose that $f(z)$ is analytic on the punctured plane $D = \mathbb{C} \setminus \{0\}$. Show that there is a constant c such that $f(z) - \dfrac{c}{z}$ has a primitive in D. Provide a formula for c in terms of an integral of $f(z)$.