Math 120A **Homework 3** **Due August 18, 2019**

1. Suppose that P and Q are smooth functions on the annulus $\{z \in \mathbb{C} \mid a < |z| < b\}$ that satisfy $\dfrac{\partial p}{\partial y} = \dfrac{\partial Q}{\partial x}$. Show directly using Green's theorem that $\displaystyle\oint_{|z|=r} P\,dx + Q\,dy$ is independent of the radius r, for $a < r < b$.

2. Show that if D is a bounded domain with smooth boundary, then
$$\int_{\partial D} \bar{z}\,dz = 2i\,\mathrm{Area}(D).$$

3. Suppose $h(z)$ is a continuous function on a curve γ. Show that
$$H(w) = \int_\gamma \frac{h(z)}{z-w}\,dz, \quad w \in \mathbb{C} \setminus \gamma,$$
is analytic on the complement of γ, $\mathbb{C} \setminus \gamma$. Find $H'(w)$.

4. Show that an analytic function $f(z)$ has a primitive in a domain D if and only if $\displaystyle\int_\gamma f(z)\,dz = 0$ for every closed path γ in D.

5. By integrating $e^{-\frac{z^2}{2}}$ around a rectangle $\mathcal{R}_t$ with vertices $-R$, R, $R+it$, $-R+it$, and sending $R \to \infty$; show that
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, e^{-itx}\,dx = e^{-\frac{t^2}{2}}, \quad -\infty < t < \infty.$$
Use the known value of the integral for $t = 0$.

   Remark: This shows that the Fourier transform of $e^{-\frac{x^2}{2}}$ is $e^{-\frac{t^2}{2}}$, a fundamental result in the study of the Fourier transform.

6. Suppose $f(z)$ is continuous in the closed disk $\{z \mid |z| \le R\}$ and analytic on the open disk $\{z \mid |z| < R\}$. Show that $\displaystyle\oint_{|z|=R} f(z)\,dz = 0$.
[Hint: Approximate $f(z)$ uniformly by $f_r(z) := f(rz)$. Why can't we just use Green's theorem?]

7. Show that a harmonic function is C^∞; that is, a harmonic function has partial derivatives of all orders.

8. Use the Cauchy integral formula to derive the mean value property of harmonic functions:
$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(z_0 + \rho e^{i\theta}\right)\,d\theta, \quad z_0 \in D,$$
whenever $u(z)$ is harmonic in the domain D and the closed disk $|z - z_0| \le \rho$ is contained in D.

9. Show that the series
$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$
converges.
[Hint: Show that the partial sums of the series satisfy $S_2 < S_4 < S_6 < \cdots < S_5 < S_3 < S_1$.]

10. Show that

    (a) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ \log(k)}$ diverges.

    (b) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ (\log(k))^2}$ converges.

11. Show that $\displaystyle\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ converges uniformly for $|z| < 1$.

12. Show that

    (a) $\displaystyle\sum_{k=1}^{\infty} \frac{z^k}{k}$ does not converge uniformly for $|z| < 1$.

    (b) $\displaystyle\sum_{k=1}^{\infty} \frac{z^k}{k}$ converges uniformly for $|z| \leq \rho$ whenever $0 < \rho < 1$.