1. Let $S \subset \mathbb{C}$ be the set $S = \{z = -it \mid t \geq 0\}$. Define explicitly a continuous branch of $\log(z)$ on $\mathbb{C} \setminus S$; that is, the complex plane slit along the negative imaginary axis. (Hint: Recall that $\text{Log}(z)$, the principal branch of $\log(z)$, is explicitly defined as $\text{Log}(z) = \log|z| + i\theta$, with $-\pi < \theta < \pi$. Be sure to explain why the strict inequalities are required for continuity.)

2. Determine the value(s) of $\log\left[(1+i)^{2i}\right]$.

3. (a) Set $w = \cos(z)$ and $\zeta = e^{iz}$. Show that $\zeta = w + \sqrt{w^2 - 1}$, where $\sqrt{w^2-1} = \left\{\xi \mid \xi^2 = w^2 - 1\right\}$. How many elements are in $\sqrt{w^2-1}$ and how are they related?

   (b) Show that
$$\cos^{-1}(w) = -i \log\left[w + \sqrt{w^2 - 1}\right],$$
   where both sides of the identity are to be interpreted as subsets of $\mathbb{C}$.

4. Let $h : [0,1] \to \mathbb{C}$ be a continuous complex-valued function $h(t)$ on the unit interval. Define
$$H(z) := \int_0^1 \frac{h(t)}{t - z}\, dt, \quad z \in \mathbb{C} \setminus [0,1]$$
   Show that $H(z)$ is analytic on $\mathbb{C} \setminus [0,1]$ and compute its derivative. (Hint: Differentiate 'by hand;' that is, by using the limit definition of the complex derivative.)

5. Given an analytic function $f(z) = u(z) + iv(z)$ (with u, v real-valued), we can write z in polar form $z = re^{i\theta}$ and define $u(r,\theta) := u\left(re^{i\theta}\right)$ and $v(r,\theta) := v\left(re^{i\theta}\right)$.

   (a) Derive the polar form of the Cauchy-Riemann equations for u and v:
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$
   (If you are studying a physical science, you should check that this definition yields consistent units across both equations.)

   (b) Check that for any integer m, the functions $u(r,\theta) = r^m \cos(m\theta)$ and $v(r,\theta) = r^m \sin(m\theta)$ satisfy (the polar form of) the Cauchy-Riemann equations.

6. Recall that $\cos^{-1}(z) = -i \log\left[z + \sqrt{z^2-1}\right]$ (with $\sqrt{z^2-1} = \left\{w \mid w^2 = z^2 - 1\right\}$). Suppose $g(z)$ is an analytic branch of $\cos^{-1}(z)$, defined on a domain D.

   (a) Determine $g'(z)$.

   (b) Do different branches of $\cos^{-1}(z)$ have the same derivative?

7. Let $f(z)$ be a bounded analytic function, defined on a bounded domain D in the complex plane, and suppose that $f(z)$ is one-to-one. Show that the area of $f(D)$ is given by
$$\text{Area}\,(f\,(D)) = \iint_D \left|f'(z)\right|^2\, dx\, dy$$
   (Hint: Review the change of variables theorem from your vector calculus course.)

8. Show that Laplace's equation in polar coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

   (If you are studying a physical science, you should check that this definition yields consistent units for each term in the equation.)

9. (a) Using Laplace's equation in polar coordinates, show that $u(r,\theta) = \theta \log(r)$ is harmonic. (Remember that $r > 0$.)

   (b) Using polar form of the Cauchy-Riemann equations, find a harmonic conjugate $v(r,\theta)$ for $u(r,\theta)$. What is the corresponding analytic function $f(z) = u(z) + iv(z)$?

10. Given a positive number B with $0 < B < \pi$. Find a conformal map of the wedge $\{z \mid -B < \arg(z) < B\}$ onto the right half-plane $\{w \mid \mathrm{Re}(w) > 0\}$.

11. The inversion mapping $f(z) = \dfrac{1}{z}$ is defined on the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ by defining $f(0) = \infty$, $f(\infty) = 0$, and $f(1) = 1$. (Remember that this does ***not*** mean that ∞ is a complex number.)

    (a) Show that the image of a straight line under f is a circle or a straight line, depending on whether or not the line passes through the origin.

    (b) Show that the image of a circle under f is a straight line or a circle, depending on whether or not the circle passes through the origin.

    Notice that both circles and straight lines in the complex plane $\mathbb{C}$ correspond to circles on the Riemann sphere S.

12. Let $P(x,y)$ and $Q(x,y)$ be continuous complex-valued functions on a curve $\boldsymbol{\gamma}$, and define a function $F(w)$ on $\mathbb{C} \setminus \boldsymbol{\gamma}$ by

$$F(w) := \int_{\boldsymbol{\gamma}} \frac{P(x,y)}{z-w}\,dx + \int_{\boldsymbol{\gamma}} \frac{Q(x,y)}{z-w}\,dy, \quad \text{where } z = x + iy.$$

    (a) Show that $F(w)$ is analytic for $w \in \mathbb{C} \setminus \boldsymbol{\gamma}$.

    (b) Express $F'(w)$ as a line integral over $\boldsymbol{\gamma}$.