Math 142B **Final Examination v. A** **Name:** ____________
September 8, 2018 **(Total Points: *50*)** **PID:** ____________

Instructions

1. **Write your Name and PID in the spaces provided above.**
2. **Make sure your Name is on every page.**
3. **No calculators, tablets, phones, or other electronic devices are allowed during this exam.**
4. **Put away ANY devices that can be used for communication or can access the Internet.**
5. **You may use one handwritten page of notes, but no books or other assistance during this exam.**
6. **Read each question carefully and answer each question completely.**
7. **Write your solutions clearly in the spaces provided.**
8. **Show all of your work. No credit will be given for unsupported answers, even if correct.**

(2 points) 0. Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

(8 points) 1. Consider the function $f : [0,1] \to \mathbb{R}$ given by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ -1 & \text{if } x \text{ is irrational.} \end{cases}$

(a) Is f integrable? Justify your answer.

(b) Is $|f|$ integrable? Justify your answer. (Note: $|f|$ is defined by $|f|(x) = |f(x)|$.)

(8 points) 2. Suppose that $f : [a, b] \to \mathbb{R}$ is a Lipschitz function; that is, there is a constant $C > 0$ such that $|f(u) - f(v)| \le C\,|u - v|$ for all $u, v \in [a, b]$.

(a) Show that $0 \le U(f, P) - L(f, P) \le C(b - a) \cdot \operatorname{gap}(P)$ for every partition P of $[a, b]$.

(b) Show that f is integrable.

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(8 points) 3. Consider the sequence of functions $\{f_n : \mathbb{R} \to \mathbb{R}\}$ defined by $f_n(x) = \frac{1}{n}\tan^{-1}\left(n^2 x\right)$.

(a) Determine the pointwise limit function $f(x) = \lim_{n\to\infty} f_n(x)$.

(b) Show that $\{f_n'(0)\}$ is unbounded.

(c) Does $\{f_n\}$ converge uniformly to f? Justify your answer.

(8 points) 4. Consider the sequence of functions $\{f_n : (-1,1) \to \mathbb{R}\}$ defined by $f_n(x) = \sum_{k=0}^{n} x^k$.

(a) Show that $\{f_n\}$ converges pointwise on $(-1,1)$ to $f(x) = \dfrac{1}{1-x}$ and verify that the convergence is not uniform.

(b) Show that $\{f_n\}$ converges uniformly to f on $[-\rho, \rho]$ for every ρ such that $0 < \rho < 1$.

(8 points) 5. Integrating both sides of the algebraic identity $\dfrac{1}{1+t^2} = \sum_{k=0}^{n}(-1)^k t^{2k} + \dfrac{(-1)^{n+1}t^{2n+2}}{1+t^2}$ implies that for every $n \in \mathbb{N}$ and every x, $\tan^{-1}(x) = \sum_{k=0}^{n} \dfrac{(-1)^k}{2k+1}x^{2k+1} + (-1)^{n+1}\displaystyle\int_0^x \dfrac{t^{2n+2}}{1+t^2}\,dt.$

Prove that for $|x| < 1$,

$$\tan^{-1}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}x^{2k+1}.$$

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(8 points) 6. Write the letter of the sequence in the right column in the space next to the matching description in the left column. No justification is required. Yes, there is an extra sequence that will not be matched. Each correctly placed letter (including the unplaced letter) will earn one point.

____	A sequence of continuous functions $\{f_n\}$ whose pointwise limit function f is not continuous	**A.**	$f_n : \mathbb{R} \to \mathbb{R}$ $f_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k$
____	A sequence of continuous functions $\{f_n\}$ whose pointwise limit function f is continuous but $\int_0^1 f \neq \lim_{n\to\infty} \int_0^1 f_n$	**B.**	$f_n : [0,1] \to \mathbb{R}$ $f_n(x) = \begin{cases} 1 & \text{if } x = \frac{k}{2^n}, k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$
____	A sequence of polynomial functions $\{f_n\}$ whose pointwise limit function f is analytic	**C.**	$f_n : [0,1] \to \mathbb{R}$ $f_n(x) = x^n$
____	A sequence of integrable functions $\{f_n\}$ whose pointwise limit function f is not integrable	**D.**	$f_n : \mathbb{R} \to \mathbb{R}$ $f_n(x) = \frac{1}{n} \tan^{-1}\left(n^2 x\right)$
____	A sequence of infinitely differentiable functions $\{f_n\}$ whose uniform limit function f is not differentiable	**E.**	$f_n : [0,1] \to \mathbb{R}$ $f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x < \frac{1}{n} \\ 2n - n^2 x & \text{if } \frac{1}{n} \le x < \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} \le x \le 1 \end{cases}$
____	A sequence of bounded functions $\{f_n\}$ whose pointwise limit function f is unbounded	**F.**	$f_n : (-1,1) \to \mathbb{R}$ $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$
____	A sequence of noncontinuous functions $\{f_n\}$ whose uniform limit function f is continuous	**G.**	$f_n : [0,1] \to \mathbb{R}$ $f_n(x) = \dfrac{x}{x^2 + \frac{1}{n}}$
		H.	$f_n : [-1,1] \to \mathbb{R}$ $f_n(x) = \begin{cases} -\frac{1}{n} & \text{if } -1 \le x < 0 \\ \frac{1}{n} & \text{if } 0 \le x \le 1 \end{cases}$