Math 102 Homework Assignment 7
Due Thursday, June 3, 2021

1. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

   (a) Determine the singular values of A.

   (b) Find an orthogonal matrix V so that $V^T A^T A V = \Sigma^2 = \begin{pmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{pmatrix}$ with $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

   (c) Find an orthogonal matrix U such that for each nonzero singular value σ_i of A, the corresponding columns $\mathbf{u}_i$ of U and $\mathbf{v}_i$ of V satisfy $\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i$.

   (d) Write the singular value decomposition $A = U\Sigma V^T$.

2. Let $A = U\,\Sigma\,V^T$ be the singular value decomposition of a $m \times n$ matrix A of rank r with nonzero singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. Write $U = \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_m \end{pmatrix}$ and $V = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix}$.

   (a) Show that $\begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_r \end{pmatrix}$ is an orthonormal basis for $R(A)$.

   (b) Show that $\begin{pmatrix} \mathbf{u}_{r+1} & \cdots & \mathbf{u}_m \end{pmatrix}$ is an orthonormal basis for $N(A^T)$.

   (c) Show that $\begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_r \end{pmatrix}$ is an orthonormal basis for $R(A^T)$.

   (d) Show that $\begin{pmatrix} \mathbf{v}_{r+1} & \cdots & \mathbf{v}_n \end{pmatrix}$ is an orthonormal basis for $N(A)$.

3. Show that if A is a symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the singular values of A are $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$.

4. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$.

   Determine the pseudoinverse A^+ and verify that A and A^+ satisfy the four Penrose conditions:

   1. $A\,A^+A = A$.
   2. $A^+A\,A^+ = A^+$.
   3. $(A\,A^+)^T = AA^+$.
   4. $(A^+A)^T = A^+A$.

5. Given a $m \times n$ matrix A, and let A^+ be the pseudoinverse of A. Verify each of the following identities.

   (a) $(A^+)^+ = A$.

   (b) $(A\,A^+)^2 = A\,A^+$.

   (c) $(A^+A)^2 = A^+A$.

6. Show that if A is a $m \times n$ matrix of rank n, then the pseudoinverse $A^+ = \left(A^T A\right)^{-1} A^T$.