**Math 102 Homework Assignment 6**
**Due Thursday, May 20, 2021**

1. Let $A$ be a $n \times n$ matrix with real entries and let $\lambda_1 = a + bi$ (where $a$ and $b$ are real and $b \neq 0$) be an eigenvalue of $A$. Let $\mathbf{z}_1 = \mathbf{x} + i\mathbf{y}$ (where $\mathbf{x}$ and $\mathbf{y}$ both have real entries) be an eigenvector of $A$ corresponding to $\lambda_1$, and let $\mathbf{z}_2 = \mathbf{x} - i\mathbf{y}$.
   (a) Explain why $\mathbf{z}_1$ and $\mathbf{z}_2$ must be linearly independent.
   (b) Show that $\mathbf{y} \neq \mathbf{0}$ and that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

2. Let $\mathbf{x}$, $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$ with $n \geq 2$, and let $A = \mathbf{x}\mathbf{y}^T$. Show that
   (a) $\lambda = 0$ is an eigenvalue of $A$ with $n-1$ linearly independent eigenvectors and, consequently, has multiplicity at least $n - 1$.
   (b) the remaining eigenvalue of $A$ is
   $$\lambda_n = \text{tr}(A) = \mathbf{x}^T\mathbf{y},$$
   and $\mathbf{x}$ is an eigenvector corresponding to $\lambda_n$.
   (c) if $\lambda_n = \mathbf{x}^T\mathbf{y} \neq 0$, then $A$ is diagonalizable.

3. Show that $e^A$ is nonsingular for every $n \times n$ matrix $A$. [Hint: Use the definition of $e^{tA}$ as the solution to a matrix initial value problem.]

4. Let $A$ be a $n \times n$ Hermitian matrix and let $\mathbf{x}$ be a vector in $\mathbf{C}^n$. Show that $\mathbf{x}^H A\mathbf{x}$ is real.

5. Let $U$ be a unitary matrix. Prove that
   (a) $U$ is normal.
   (b) $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{C}^n$.
   (c) if $\lambda$ is an eigenvalue of $U$, then $|\lambda| = 1$.

6. Let $A$ be a Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Show that
$$A = \lambda_1\mathbf{u}_1\mathbf{u}_1^H + \lambda_2\mathbf{u}_2\mathbf{u}_2^H + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^H.$$