Math 102 Homework Assignment 4
Due Thursday, May 6, 2021

1. Let P_3 be the inner product space of polynomials of degree less than 3 with inner product $\langle p, q\rangle = \int_{-1}^{1} p(t)q(t)\,dt$. Let $g_1(t) = 1$ and $g_2(t) = t$. Find a basis for $\text{Span}(g_1, g_2)^{\perp}$, the orthogonal complement of the subspace of P_3 spanned by g_1 and g_2.

2. Let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors in an inner product space V.
   (a) Show that $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.
   (b) Show that if $\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

3. Let $A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.
   (a) Find the projection matrix P that projects vectors in $\mathbb{R}^4$ onto $R(A)$.
   (b) Find an orthonormal basis for $N(A^T)$.
   (c) Determine the projection matrix Q that projects vectors in $\mathbb{R}^4$ onto $N(A^T)$.

4. Let A be a $m \times n$ matrix, let P be the projection matrix that projects vectors in $\mathbb{R}^m$ onto $R(A)$, and let Q be the projection matrix that projects vectors in $\mathbb{R}^n$ onto $R(A^T)$.
   (a) Show that $I - P$ is the projection matrix from $\mathbb{R}^m$ onto $N(A^T)$.
   (b) Show that $I - Q$ is the projection matrix from $\mathbb{R}^n$ onto $N(A)$.

5. Let $\mathbf{v}$ be a vector in an inner product space V and let $\mathbf{p}$ be the projection of $\mathbf{v}$ onto an n-dimensional subspace S of V. Show that $\|\mathbf{p}\|^2 = \langle \mathbf{p}, \mathbf{v}\rangle$.

6. Consider the inner product space $C[-1, 1]$ of continuous functions on $[-1, 1]$ with inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt.$$

   Find an orthonormal basis for $\text{Span}(1, x, x^2)$, the subspace of $C[-1, 1]$ spanned by $\{1,\ x,\ x^2\}$.

7. Determine the QR factorization of $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

8. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n\}$ be an orthonormal basis for an inner product space V. Let $S_1 = \text{Span}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$, the subspace of V spanned by $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ and $S_2 = \text{Span}(\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n)$, the subspace of V spanned by $\{\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n\}$. Show that $S_1 \perp S_2$.