Math 102 Homework Assignment 3
Due Thursday, April 22, 2021

1. Let A be a $m \times n$ matrix. Show that:
   (a) If $\mathbf{x} \in N\left(A^T A\right)$, then $A\mathbf{x} \in R(A) \cap N\left(A^T\right)$.
   (b) $N\left(A^T A\right) = N(A)$.

2. Let V and W be subspaces of $\mathbb{R}^n$ such that $V \subset W$. Show that $W^\perp \subset V^\perp$.

3. Suppose A is a symmetric $n \times n$ matrix. Let V be a subspace of $\mathbb{R}^n$ with the property that $A\mathbf{x} \in V$ for every $\mathbf{x} \in V$. Show that $A\mathbf{y} \in V^\perp$ for every $\mathbf{y} \in V^\perp$. (Remark: The subspace V is said to be invariant under A. This exercise shows that if V is an invariant subspace under A, then so is $V^\perp$.)

4. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}$.
   (a) Find the orthogonal projection of $\mathbf{b}$ onto $R(A)$.
   (b) Describe all least squares solutions to $A\mathbf{x} = \mathbf{b}$.

5. Let A be a $n \times n$ matrix. Given that $\begin{pmatrix} A & I \\ O & A^T \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix}$.
   Show that $\hat{\mathbf{x}}$ is a least squares solution of the system $A\mathbf{x} = \mathbf{b}$ and that $\mathbf{r}$ is the residual vector.

6. Given a $m \times n$ matrix A. Let $\hat{\mathbf{x}}$ be a solution to the least squares problem $A\mathbf{x} = \mathbf{b}$. Show that a vector $\mathbf{y} \in \mathbf{R}^n$ will also be a least squares solution if an only if $\mathbf{y} = \hat{\mathbf{x}} + \mathbf{z}$ for some vector $\mathbf{z} \in N(A)$.