Math 102 Homework Assignment 2
Due Thursday, April 15, 2021

1. Let L be a linear operator on $\mathbb{R}^n$ with the property that $L(\mathbf{x}) = \mathbf{0}$ for some nonzero vector $\mathbf{x} \in \mathbb{R}^n$. Let $A = \big[L\big]_{\mathcal{E}}$ be the matrix representing L with respect to the standard basis $\mathcal{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$. Show that A is singular.

2. Let L be a linear operaton on a vector space V. Let $A = \big[L\big]_{\mathcal{B}}$ be the matrix representing L with respect to a basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V. Show that A^m is the matrix representing L^m with respect to $\mathcal{B}$; that is, show that $\big[L^m\big]_{\mathcal{B}} = \Big(\big[L\big]_{\mathcal{B}}\Big)^m$.

3. Suppose $A = S\Lambda S^{-1}$, where Λ is a diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_n$. Write $S = \begin{bmatrix}\mathbf{s}_1 & \cdots & \mathbf{s}_n\end{bmatrix}$; that is, $\mathbf{s}_i$ is the i^{th} column of S.
   (a) Show that $A\mathbf{s}_i = \lambda_i \mathbf{s}_i$ for $i = 1, \ldots, n$.
   (b) Show that if $\mathbf{x} = \alpha_1\mathbf{s}_1 + \alpha_2\mathbf{s}_2 + \cdots + \alpha_n\mathbf{s}_n$, then $A^k\mathbf{x} = \alpha_1\lambda_1^k\mathbf{s}_1 + \alpha_2\lambda_2^k\mathbf{s}_2 + \cdots + \alpha_n\lambda_n^k\mathbf{s}_n$.
   (c) Suppose $|\lambda_i| < 1$ for $i = 1, \ldots, n$. What happens to $A^k\mathbf{x}$ as $k \to \infty$? Explain.

4. Let A and B be similar matrices.
   (a) Show that A^T and B^T are similar.
   (b) Show that A^k and B^k are similar for every positive integer k.

5. Given any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Prove the triangle inequality: $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}| + \|\mathbf{v}\|$. [Hint: Show that $\|\mathbf{u} + \mathbf{v}\|^2 \leq (\|\mathbf{u}| + \|\mathbf{v}\|)^2$.]

6. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$. Define

$$\mathbf{p} = \left(\frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\right)\mathbf{y} \quad \text{and} \quad \mathbf{z} = \mathbf{x} - \mathbf{p}.$$

   (a) Show that $\mathbf{p} \perp \mathbf{z}$.
   (b) If $\|\mathbf{p}\| = 6$ and $\|\mathbf{z}\| = 8$, determine the value of $\|\mathbf{x}\|$.