Math 142B
Summer 2011 Midterm Exam 1 Solution

1. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Show the f is integrable on $[0,1]$ and determine the value of $\int_0^1 f$.

Given a natural number n, $f(x) = 0$ for each x in $(\frac{1}{k}, \frac{1}{k-1})$ for all integers $k = 2, \ldots, n$. Thus, f is a step function, hence integrable, on $[\frac{1}{n}, 1]$. Choose $P_n^\star$ a partition of $[\frac{1}{n}, 1]$ such that $U(f, P_n^\star) - L(f, P_n^\star) < \frac{1}{n}$, and let $P_n = P_n^\star \cup \{0\}$. Then, P_n is a partition of $[0,1]$ and

$$U(f, P_n) - L(f, P_n) = U(f, P_n^\star) - L(f, P_n^\star) + U(f, \{0, \frac{1}{n}\}) - L(f, \{0, \frac{1}{n}\}) < \frac{1}{n} + \frac{1}{n} \to 0.$$

It follows that $\{P_n\}$ is Archimedean for f on $[0,1]$. Hence, f is integrable. Since $L(f, P) = 0$ for every partition P of $[0,1]$, $\int_0^1 f = 0$.

2. Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that $\int_c^d f \geq 0$ for all c, d with $a \leq c < d \leq b$. Prove that $f(x) \geq 0$ for all $x \in [a,b]$.

Suppose $f(x_0) = -\rho < 0$ at some x_0 in $[a,b]$. Since f is continuous, there is a $\delta > 0$ such that $|f(x) - f(x_0)| = |f(x) + \rho| < \frac{\rho}{2}$ for all x such that $|x - x_0| < \delta$. Thus, $-\frac{3\rho}{2} < f(x) < -\frac{\rho}{2} < 0$ for all x in $(x_0 - \delta, x_0 + \delta)$. It follows that $\int_{x_0-\delta}^{x_0+\delta} f < 0$.

3. Exhibit an example of a function $f : [0,1] \to \mathbb{R}$ that is unbounded.

The function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0 \end{cases}$$

is unbounded since $\lim_{x \to 0^+} \frac{1}{x}$ diverges to infinity.

4. For numbers $a_1, \ldots, a_n$, define $p(x) = a_1 x + a_2 x^2 + \cdots + a_n x^n$ for all x. Suppose that

$$\frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0.$$

Prove that there is an $x_0 \in (0,1)$ such that $p(x_0) = 0$.

By the Fundamental Theorem of Calculus, $\int_0^1 p = \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1}$.

Suppose $p(x) \neq 0$ for all x in $(0,1)$. Then, $p(x) > 0$ for all x in $(0,1)$ or $p(x) < 0$ for all x in $(0,1)$, since p is continuous. If $p(x) > 0$ for all x in $(0,1)$, then $\int_0^1 p > 0$. Similarly, if $p(x) < 0$ for all x in $(0,1)$, then $\int_0^1 p < 0$. It follows that $\frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} \neq 0$.

5. Let $f : [a, b] \to \mathbb{R}$ be monotonically increasing.

(a) Show that f is bounded on $[a, b]$.

$f(a) \leq f(x) \leq f(b)$ for all x in $[a, b]$, since f is monotonically increasing.

(b) Let P_n be a regular partition of $[a, b]$ into n partition intervals. Show that

$$U(f, P_n) - L(f, P_n) = \frac{[f(b) - f(a)][b - a]}{n}$$

$$\begin{aligned}
U(f, P_n) - L(f, P_n) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} (M_i - m_i)\frac{[b - a]}{n} \quad \text{since } P_n \text{ is a regular partition} \\
&= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\frac{[b - a]}{n} \quad \text{since } f \text{ is monotonically increasing} \\
&= \frac{[f(b) - f(a)][b - a]}{n}.
\end{aligned}$$