Final Exam

Follow these instructions carefully.

1. No calculators or other electronic computational aids may be used during the exam.

2. You may have one page of notes, but no books or other assistance.

3. Write your name, PID, and section on the cover of your bluebook.

4. Show all your work in the bluebook.

5. No credit will be given for unsupported answers.

6. Present your answer clearly.

   a) Carefully indicate the number and letter of each question and question part.

   b) Try to present your answers in the same order they appear in the exam.

   c) Start each question on a new side of a page.

There are six questions. Questions 1-4 are worth 10 points, questions 5 and 6 are worth 20 points.

Question 1.

Make a truth table for the following Boolean formula:

$$(p \lor q) \to (\neg q \land p)$$

Is the formula a tautology?

Question 2.

Let $P(x)$ and $Q(x)$ be predicates. Find statements that are equivalent to the negation of each of the following sentences. Write your answers so that no quantifier is in the scope of a negation. Show all your work.

(a) $(\forall x)(\exists y)(P(x) \to Q(y))$

(b) $((\exists x)P(x)) \land (\exists x)(Q(x))$.

Question 3.

Let X be the set of subsets of $\{1, 2, 3, 4\}$ of size at least 2. The subset relation is a partial order on X. Extend this partial order to a total order on X.

Question 4.

Let $X = \{1, 2, 3\}$, and $Y = \{1, 2, 3, 4, 5\}$.

(a) How many injections are there from X to Y?

(b) How many surjections are there from Y to X?

(c) How many functions are there from X to Y?

Question 5.

(a) Use the division theorem to prove that if $3|n^2$ then $3|n$. *Hint*: show the contrapositive. Split the argument into two cases depending on the remainder of dividing n by 3.

(b) Give a proof by contradiction that $\sqrt{3}$ is irrational. *Hint*: give a proof by contradiction. Assume that $\sqrt{3} = a/b$ where a and b are integers, and the fraction a/b is in lowest terms.

Question 6.

In this question, we will prove the following version of the pigeonhole principle: For all $n \geq 2$, if $f : \{1, \ldots, n\} \to \{1, \ldots, n-1\}$ is a function, then f is not injective. Consider the following predicate:

$$P(n) = (\forall f)(\text{if } f : \{1, \ldots, n\} \to \{1, \ldots, n-1\} \text{ is a function, then } f \text{ is not injective})$$

We will prove the claim by weak induction on n.

(a) The base case is $n = 2$. Write out all possible functions from $\{1, 2\}$ to $\{1\}$.

(b) Show that $P(2)$ is true.

(c) Fix $n > 2$. If $f : \{1, \ldots, n\} \to \{1, \ldots, n-1\}$ is function, define the function $g : \{1, \ldots, n-1\} \to \{1, \ldots, n-2\}$ as follows:

$$g(x) = \begin{cases} f(x) & \text{if } f(x) < n-1 \\ f(n) & \text{if } f(x) = n-1 \end{cases}$$

Argue that if f is an injection, then g is an injection.

(d) For the induction step, show that for $n > 2$ that $P(n-1)$ implies $P(n)$ by showing the contrapositive.