Complex Analysis Qualifying Exam Solutions

May 21, 2014

Part 1.

1. Let $\log z$ be the principal branch of the logarithm defined on $G = \{z \in \mathbb{C} \,|\, z \notin (-\infty, 0]\}$. Show that if $t > 0$, then the equation $\log z = \frac{t}{z}$ has exactly one root in G.

Solution. Let $z = re^{i\theta}$ where $r > 0$ and $-\pi < \theta < \pi$. Then $\log z = \frac{t}{z}$ becomes

$$\log r + i\theta = \frac{t}{r}\cos\theta - i\frac{t}{r}\sin\theta.$$

Since θ and $-\frac{t}{r}\sin\theta$ have opposite signs when $\theta \neq 0$, it follows that if $z \in G$, then $\log z = \frac{t}{z}$ if and only if $z = r$ where $r > 0$ satisfies $\log r = \frac{t}{r}$. But $\log r$ is strictly increasing and $\frac{t}{r}$ is strictly decreasing. Hence, the Intermediate Value Theorem implies that $\log r = \frac{t}{r}$ has exactly one root in $\{r \in \mathbb{R} \,|\, r > 0\}$. Therefore, $\log z = \frac{t}{z}$ has exactly one root in G.

2. Let $G = \mathbb{C} \setminus [-1, 1]$. Prove that $f(z) = z^2 - 1$ has an analytic square root on G but does not have an analytic logarithm on G.

Solution. To see that f has a square root, consider the function g defined on G by the formula

$$g(z) = |f(z)|^{\frac{1}{2}}\, e^{\frac{1}{2}\left(\arg(z-1)+\arg(z+1)\right)},$$

where the arguments are chosen in $[0, 2\pi)$. g is a well defined continuous function on G satisfying $g^2 = f$. (g is analytic since z^2 is locally 1-1 on $\mathbb{C} \setminus \{0\}$; cf. Proposition 2.20 pg. 39 Conway).

To see that f does not have an analytic logarithm on G, recall that this is equivalent to $\frac{f'}{f}$ having a primitive on G. But if γ is a path in G that winds once around $[-1, 1]$ in the counterclockwise direction, then

$$\int_\gamma \frac{f'}{f}\, dz = \int_\gamma \left(\frac{1}{z-1} + \frac{1}{z+1}\right) dz = 4\pi i \neq 0.$$

Therefore, $\frac{f'}{f}$ cannot have a primitive (cf. Corollary 1.22 pg. 66 Conway).

3. Prove that the zeros of the polynomial $p(z) = z^n + c_{n-1}z^{n-1} + \ldots c_1 z + c_0$ all lie in the open disk with center 0 and radius

$$R = \sqrt{1 + |c_{n-1}|^2 + \ldots + |c_1|^2 + |c_0|^2}.$$

Solution. Note that $R = 1$ if and only if $p(z) = z^n$ and that in this case the assertion is obviously true. Therefore, we may assume that $R > 1$. If $|z| = R$, then

$$\begin{aligned}
|z^n - p(z)| &= |c_{n-1}z^{n-1} + \ldots c_1 z + c_0| \\
&\le (|c_{n-1}|^2 + \ldots + |c_1|^2 + |c_0|^2)^{\frac{1}{2}} (R^{2(n-1)} + R^{2(n-2)} + \ldots + 1)^{\frac{1}{2}} \\
&= \left(R^2 - 1\right)^{\frac{1}{2}} \left(\frac{R^{2n} - 1}{R^2 - 1}\right)^{\frac{1}{2}} \\
&= \left(R^{2n} - 1\right)^{\frac{1}{2}} \\
&< R^n \\
&= |z^n|.
\end{aligned}$$

Therefore, by Rouche's Theorem, the 0's of p all lie in the open disk with center 0 and radius R.

4. Let G be a connected open set in $\mathbb{C}$ and let $a \in G$. Prove that if $\mathcal{F} \subseteq \mathrm{H}(G)$ is a normal family, then $\mathcal{D} = \{f \in \mathrm{H(G)} \,|\, f(a) = 0 \text{ and } f' \in \mathcal{F}\}$ is a normal family.

Solution. Assume that $\mathcal{F} \subseteq \mathrm{H}(G)$ is normal. That $\mathcal{D}$ is normal, will follow from Montel's Theorem if we can show that $\mathcal{D}$ is locally bounded. Accordingly, fix $b \in G$.

Since G is assumed connected, there exists a rectifiable path $\gamma : [0,1] \to G$ satisfying $\gamma(0) = a$ and $\gamma(1) = b$. Since $\mathcal{F}$ is normal and $\{\gamma\}$ is compact, there exists a constant M_1 such that

$$\forall_{g \in \mathcal{F}} \max_{t \in [0,1]} |g(\gamma(t))| \le M_1.$$

Also, since $\mathcal{F}$ is normal, there exist $r > 0$ and M_2 such that $B(b, r) \subseteq G$ and

$$\forall_{g \in \mathcal{F}} \sup_{z \in B(b,r)} |g(z)| \leq M_2.$$

It follows that if $f \in \mathcal{D}$ and $w \in B(b, r)$, then

$$\begin{aligned} |f(w)| &= \Big| \int_\gamma f'(z)\, dz + \int_{[b,w]} f'(z)\, dz \Big| \\ &\leq \Big| \int_\gamma f'(z)\, dz \Big| + \Big| \int_{[b,w]} f'(z)\, dz \Big| \\ &\leq M_1 \|\gamma\| + M_2 r. \end{aligned}$$

Thus, $\mathcal{D}$ is locally bounded.

5. Show that if $G \neq \mathbb{C}$ is a simply connected subset of $\mathbb{C}$, $f : G \to G$ is analytic, and $f(z)$ is not identically equal to z, then f has at most one fixed point in G.

Solution. Recall that Problem 2 on the 220B Final Exam was to show that if $g : \mathbb{D} \to \mathbb{D}$ is analytic and $g(z)$ is not identically equal to z, then g can have at most one fixed point in $\mathbb{D}$. The simple proof of this fact was based on an application of Schwarz's Lemma.

Since G is assumed simply connected and $\neq \mathbb{C}$, it follows by the Riemann Mapping Theorem that there exists an analytic bijection $\phi : G \to \mathbb{D}$. If we define g by $g = \phi \circ f \circ \phi^{-1}$, then $g : \mathbb{D} \to \mathbb{D}$ is analytic and $f(z)$ is not identically equal to z if and only if $g(z)$ is not identically equal to z. Furthermore, $a \in G$ is a fixed point for f if and only if $\phi(a)$ is a fixed point in $\mathbb{D}$ for g. Therefore, the assertion of this problem follows from the result in the previous paragraph.

6. Show that if G is an open subset of $\mathbb{C}$, then there exist ideals in $\mathrm{H}(G)$ that are not finitely generated.

Solution. Construct an infinite sequence of distinct points $\{z_k\}_{k=1}^{\infty}$ in G that has no limit points in G. For each $n \geq 1$ define

$$\mathcal{I}_n = \{f \in \mathrm{H}(G) \,|\, f(z_k) = 0 \text{ for each } k \geq n\}.$$

It is straightforward to prove that for each $n \geq 1$, $\mathcal{I}_n$ is an ideal and also that $\mathcal{I}$, defined by

$$\mathcal{I} = \bigcup_{n=1}^{\infty} \mathcal{I}_n,$$

is an ideal.

We claim that $\mathcal{I}$ is not finitely generated. For if $f_1, f_2, \ldots, f_m$ generate $\mathcal{I}$, then, as there exists N such that $f_k \in \mathcal{I}_N$ for $k = 1, 2, \ldots, m$, it would follow that $\mathcal{I} = \mathcal{I}_N$. On the other hand, the Weierstrass Factorization Theorem implies the existence of an $f \in \mathcal{I}_{N+1}$ with $f(z_N) \neq 0$, i.e., $\mathcal{I} \neq \mathcal{I}_N$.

Part 2.

1. Let

$$G = \{z \in \mathbb{C} \mid \; |z| < 1 \text{ and } |z - \tfrac{1}{3}| > \tfrac{2}{3}\},$$
$$K = G^- = \{z \in \mathbb{C} \mid \; |z| \leq 1 \text{ and } |z - \tfrac{1}{3}| \geq \tfrac{2}{3}\},$$

and $\mathrm{A}(K)$ denote the space of continuous functions on K that are analytic on G equipped with the uniform norm on K. For the purposes of this problem, a *Laurent polynomial* ia a function of the form $\sum_{n=-N}^{N} a_n z^n$ and a *Laurent series* is an infinite sum of the form $\sum_{n=-\infty}^{\infty} a_n z^n$. Determine the truth or falsity of the following assertions. Be sure to justify your answers.

(a) The polynomials are dense in $\mathrm{H}(G)$.

(b) The polynomials are dense in $\mathrm{A}(K)$.

(c) If f is analytic on a neighborhood of K, then f can be uniformly approximated on K by a Laurent polynomial.

(d) If $f \in \mathrm{H}(G)$ then f can be represented on G by a Laurent series.

Solution to (a). This assertion is true. Fix a compact open exhaustion $\{K_n\}$ of G that has the property that $\mathbb{C} \setminus K_n$ is connected for each n. For example, the sets K_n defined by

$$K_n = \{z \in \mathbb{C} \mid \; |z| \leq \frac{n}{n+1} \text{ and } |z - \tfrac{1}{3}| \geq \tfrac{2}{3} \, \frac{n+1}{n}\}$$

have this property.

To prove that the polynomials are dense in $\mathrm{H}(G)$, fix $f \in \mathrm{H}(G)$. For each n, since $\mathbb{C} \setminus K_n$ is connected, Runge's Theorem implies that there exists a polynomial

p_n such that $|f(z) - p_n(z)| \le \frac{1}{n}$. For such a sequence of polynomials, $p_n \to f$ in $\mathrm{H}(G)$.

Solution to (b). This assertion is false. Note that the function $f(z) = \frac{1}{z}$ is in $\mathrm{A}(K)$. If $\{p_n\}$ is a sequence of polynomials and $p_n \to \frac{1}{z}$ in $\mathrm{A}(K)$, then there exists n such that

$$\max_{z \in K} |\tfrac{1}{z} - p_n(z)| < 1.$$

But if we set $f(z) = 1 - zp_n(z)$, as f is analytic on a neighborhood of $\mathbb{D}^-$, the Maximum Principle implies that

$$1 = |f(0)| \le \max_{|z|=1} |f(z)| = \max_{|z|=1} |\tfrac{1}{z} - p_n(z)| < 1.$$

This contradiction implies that the polynomials are not dense in $\mathrm{A}(K)$.

Solution to (c). This assertion is true. $\mathbb{C}_\infty \setminus K$ has two components,

$$C_1 = \{z \,|\, |z - \tfrac{1}{3}| < \tfrac{2}{3}\} \text{ and } C_2 = \{z \,|\, |z| > 1\} \cup \{\infty\}.$$

Furthermore, $0 \in C_1$ and $\infty \in C_2$. Therefore, by Runge's Theorem, if f is analytic on a neighborhood of K, f can be uniformly approximated on K by rational functions R whose only poles are at 0 and ∞. But if R is a rational function whose only poles are at 0 and ∞, then R is a Laurent polynomial.

Solution to (d). This assertion is false. A particularly simple counterexample is obtained by considering the function $f(z) = (z-a)^{-1}$ where $|a - \frac{1}{3}| = \frac{2}{3}$. f has two Laurent series representations,

$$f(z) = \sum_{n=1}^{\infty} \frac{a^{n-1}}{z^n}, \qquad |a| < |z|$$

and

$$f(z) = -\sum_{n=0}^{\infty} \frac{z^n}{a^{n+1}}, \qquad |z| < |a|.$$

If $a \ne -\frac{1}{3}, 1$, then neither of these series converge on G.

2. (a) Prove that the formula

$$f(z) = \lim_{\alpha \to 0+} \int_{\alpha}^{\frac{\pi}{2}} \frac{t^z}{\sin t}\, dt$$

defines an analytic function in $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$.

(b) Show that if f is defined as in part (a), then f has an analytic continuation to $\{z \in \mathbb{C} \mid \operatorname{Re} z > -1\} \setminus \{0\}$ with a simple pole at 0 with residue 1.

Solution. (a) Let $G = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. In Conway's language, the assertion is that the integral $\int_0^{\frac{\pi}{2}} \frac{t^z}{\sin t}\, dt$ converges uniformly in G. Fix a compact set $K \subseteq G$. Since K is compact, if we let

$$\rho = \min_{z \in K} \operatorname{Re} z,$$

then $\rho > 0$. Since $\frac{t}{\sin t}$ has a removable singularity at 0, there exists $\delta \in (0,1)$ and a constant c such that

$$0 < t < \delta \implies \left|\frac{1}{\sin t}\right| \le \frac{c}{t}.$$

If $\alpha, \beta \in (0, \delta)$ with $\alpha < \beta$ and $z \in K$, it follows that

$$\begin{aligned}
\left|\int_\alpha^\beta \frac{t^z}{\sin t}\, dt\right| &\le \int_\alpha^\beta \left|\frac{1}{\sin t}\right| t^{\operatorname{Re} z}\, dt \\
&\le \int_\alpha^\beta \frac{c}{t}\, t^\rho\, dt \\
&= \frac{c}{\rho}\, (\beta^\rho - \alpha^\rho).
\end{aligned}$$

Noting that either Morera's Theorem or Leibniz's Rule imply that $\int_\alpha^{\frac{\pi}{2}} \frac{t^z}{\sin t}\, dt$ is analytic on G when $\alpha \in (0, \frac{\pi}{2})$, this estimate implies that if we choose a sequence $\{\alpha_n\}$ in $(0, \frac{\pi}{2})$, then $\left\{ \int_{\alpha_n}^{\pi/2} \frac{t^z}{\sin t}\, dt \right\}$ is a Cauchy sequence in $\mathrm{H}(G)$. Since $\mathrm{H}(G)$ is complete, it follows that there exists $f \in \mathrm{H}(G)$ such that

$$\int_{\alpha_n}^{\frac{\pi}{2}} \frac{t^z}{\sin t}\, dt \to f \text{ in } \mathrm{H}(G).$$

Furthermore, the function f does not depend on the choice of sequence $\{\alpha_n\}$. This proves that there exists an analytic function f on G satisfying

$$f(z) = \lim_{\alpha \to 0+} \int_\alpha^{\frac{\pi}{2}} \frac{t^z}{\sin t}\, dt.$$

(b) We adapt the trick used in class to analytically continue the Riemann ζ function to the critical strip $\{z \,|\, 0 < \operatorname{Re} z < 1\}$. Let

$$f_1(z) = \int_0^{\frac{\pi}{2}} \left(\frac{1}{\sin t} - \frac{1}{t}\right) t^z \, dt$$

and

$$f_2(z) = \int_0^{\frac{\pi}{2}} \frac{1}{t} \, t^z \, dt.$$

Noting that $\frac{1}{\sin z} - \frac{1}{z}$ has a removable singularity at 0, it follows by a straightforward modification of the analysis in part (a) that f_1 is a well defined analytic function on $\{z \,|\, \operatorname{Re} z > -1\}$. Also, if $\operatorname{Re} z > 0$, then

$$f_2(z) = \frac{(\frac{\pi}{2})^z}{z}$$

by straightforward calculation. It follows that if $\operatorname{Re} z > 0$, then

$$f(z) = f_1(z) + f_2(z) = f_1(z) + \frac{(\frac{\pi}{2})^z}{z}.$$

Hence,

$$f_1(z) + \frac{(\frac{\pi}{2})^z}{z},$$

which is analytic in $\{z \in \mathbb{C} \,|\, \operatorname{Re} z > -1\} \setminus \{0\}$ with a simple pole at 0 gives the desired continuation of f.

$$\operatorname{Res}(f, 0) = \lim_{z \to 0} z\Big(f_1(z) + \frac{(\frac{\pi}{2})^z}{z}\Big) = (\frac{\pi}{2})^0 = 1.$$

3. Let G be an open set in $\mathbb{C}$ and let $(\mathscr{S}(G), \rho)$ denote the sheaf of germs of analytic functions on G.

(a) Prove that the sheaf topology on $\mathscr{S}(G)$ is Hausdorff.

(b) Let $\Gamma(t) = (\gamma(t), [f_t]_{\gamma(t)})$, $0 \leq t \leq 1$ be a function from $[0, 1]$ into $\mathscr{S}(G)$ and for each $t \in [0, 1]$ choose an open set D_t such that $\gamma(t) \in D_t \subseteq G$ and f_t is analytic on D_t. Prove that Γ is continuous if and only if γ is a path in G and $\{(f_t, D_t) \,|\, 0 \leq t \leq 1\}$ is an analytic continuation along γ.

Solution to (a). Assume that $(a, [f]_a), (b, [f]_b) \in \mathscr{S}(G)$ with $(a, [f]_a) \neq (b, [f]_b)$. Either $a \neq b$, or $a = b$ and $[f]_a \neq [f]_b$.

If $a \neq b$, choose disjoint neighborhoods $D_a, D_b \subseteq G$ of a and b respectively such that f is analytic on D_a and g is analytic on D_b. Then

$$N(f, D_a) = \{(z, [f]_z) \,|\, z \in D_a\} \text{ and } N(g, D_b) = \{(z, [g]_z) \,|\, z \in D_b\}$$

are disjoint neighborhoods of $(a, [f]_a)$ and $(b, [f]_b)$ respectively.

Solution to (b). First assume that $\Gamma(t) = (\gamma(t), [f_t]_{\gamma(t)})$, $0 \leq t \leq 1$ is a continuous function from $[0, 1]$ into $\mathscr{S}(G)$ and that D_t is an open set such that $\gamma(t) \in D_t \subseteq G$ and f_t is analytic on D_t. We wish to show γ is a path in G and $\{(f_t, D_t) \,|\, t \in [0, 1]\}$ is an analytic continuation along γ. Since $\gamma = \rho \circ \Gamma$, that γ is a path in G follows immediately follows from the continuity of Γ and ρ. To prove that $\{(f_t, D_t) \,|\, t \in [0, 1]\}$ is an analytic continuation along γ, fix $t \in [0, 1]$. Since Γ is continuous and $N(f_t, D_t)$ is a neighborhood of $\Gamma(t)$, there exists $\delta > 0$ such that

$$s \in [0, 1] \text{ and } |s - t| < \delta \implies \Gamma(s) \in N(f_t, D_t).$$

But $\Gamma(s) = (\gamma(s), [f_s]_{\gamma(s)})$ and $N(f_t, D_t) = \{(z, [f_t]_z) \,|\, z \in D_t\}$. Therefore,

$$s \in [0, 1] \text{ and } |s - t| < \delta \implies \gamma(s) \in D_t \text{ and } [f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}.$$

This proves that $\{(f_t, D_t) \,|\, t \in [0, 1]\}$ is an analytic continuation along γ.

Conversely, assume that γ is a path in G, $\Gamma(t) = (\gamma(t), [f_t]_{\gamma(t)})$, $0 \leq t \leq 1$ is a function from $[0, 1]$ into $\mathscr{S}(G)$, and that $\{(f_t, D_t) \,|\, t \in [0, 1]\}$ is an analytic continuation along γ. We wish to show that Γ is continuous. Accordingly, fix $t \in [0, 1]$ and an open set Ω in $\mathscr{S}(G)$ such that $\Gamma(t) \in \Omega$.

By the definition of the sheaf topology, there exists an open set U in G such that $\Gamma(t) \in N(f_t, U) \subseteq \Omega$. Since $\{(f_t, D_t) \,|\, t \in [0, 1]\}$ is an analytic continuation along γ, there exists $\delta_1 > 0$ such that

$$s \in [0, 1] \text{ and } |s - t| < \delta_1 \implies \gamma(s) \in D_t \text{ and } [f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}.$$

Since γ is a path, there exists $\delta_2 > 0$ such that

$$s \in [0, 1] \text{ and } |s - t| < \delta_2 \implies \gamma(s) \in U.$$

Therefore, if $\delta = \min\{\delta_1, \delta_2\}$, then

$$s \in [0, 1] \text{ and } |s - t| < \delta \implies \gamma(s) \in D_t \cap U \text{ and } [f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}.$$

But $\gamma(s) \in D_t \cap U$ and $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$ imply that

$$\Gamma(s) = (\gamma(s), [f_s]_{\gamma(s)}) \in \{(z, [f_t]_z) \,|\, z \in D_t \cap U\} = N(f_t, U) \subseteq \Omega.$$

Summarizing, we have shown that if $t \in [0, 1]$ and Ω is a neighborhood of $\Gamma(t)$ in $\mathscr{S}(G)$, then there exists $\delta > 0$ such that $\Gamma(s) \in \Omega$ whenever $s \in [0, 1]$ and $|s - t| < \delta$. Therefore, Γ is continuous.