Circle valued Morse theory

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In 1982 Novikov was led by physics to initiate the Morse theory of circle-valued functions, and more generally closed 1-forms. The basic problem is to determine the minimum number of critical points within a homotopy class. Various topologists have worked in the field since then, including Farber, Pazhitnov, Latour, Hutchings and Lee, as well as the speaker. The subject has grown in importance, especially in view of the connections with Floer theory and Seiberg-Witten theory. However, the talk will only be on the algebraic topology of a Morse function $f : M \to S^1$ on a compact finite-dimensional manifold M. The *Novikov complex* $C^{Nov}(M, f)$ is a f.g. free $\widehat{\mathbb{Z}[\pi_1(M)]}$-module chain complex with one generator for each critical point of f, and differentials counting the gradient flow lines of the real-valued Morse function $\overline{f} : \overline{M} \to \mathbb{R}$ on the infinite cyclic cover $\overline{M} = f^*\mathbb{R}$ of M. The *Novikov ring* $\widehat{\mathbb{Z}[\pi_1(M)]}$ is a completion of $\mathbb{Z}[\pi_1(M)]$. The talk will describe the algebraic construction and properties of the *rational Novikov complex* $C^{rat}(M, f)$ over a noncommutative localization $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ of $\mathbb{Z}[\pi_1(M)]$ inverting a set of matrices Σ. The complex $C^{rat}(M, f)$ determines $C^{Nov}(M, f)$, and is closer to the topology than $C^{Nov}(M, f)$. Originally $C^{rat}(M, f)$ was obtained (with Farber) by a perturbation of the $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$-coefficient Morse complex of a fundamental domain of $\overline{M}$. Subsequently $C^{rat}(M, f)$ was obtained by an algebraic model for the gradient flow lines of $\overline{f}$. The minimum number of critical points of a Morse function homotopic to f is conjectured to be the minimum number of generators of a chain complex in an appropriately simple chain homotopy type of $C^{rat}(M, f)$.