Math 102
Winter '08
Homework #7

$\boxed{4.3}$

3) a) True: $|S^{-1}AS| = |S^{-1}||A||S| = \frac{1}{|S|}|A|\cdot|S| = |A|$

b) False: Let $A = \begin{pmatrix}1&1\\1&1\end{pmatrix}$. Then $|A|=0$ but none of the cofactors are 0

c) False: Let $A = \begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix} \Rightarrow |A| = 1\begin{vmatrix}1&1\\0&1\end{vmatrix} - 1\begin{vmatrix}0&1\\1&1\end{vmatrix} = 1(1)-1(-1)$

$= 2$

5) Let $F_n = \begin{vmatrix}1 & -1 & & 0\\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & -1\\ 0 & & 1 & 1\end{vmatrix}$ Tridiagonal matrix.

Claim: $\{F_n\}$ is the Fibonacci sequence!

Pf: $F_1 = |1|$, $F_2 = \begin{vmatrix}1&-1\\1&1\end{vmatrix} = 1-(-1)(1) = 2$.

so the 1st two terms match up. So, it remains to show that $\{F_n\}$ satisfies the rule:

$$F_n = F_{n-1} + F_{n-2}.$$

$$F_n = \underbrace{\begin{vmatrix}1 & -1 & & 0\\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & -1\\ 0 & & 1 & 1\end{vmatrix}}_{(n)} = 1\underbrace{\begin{vmatrix}1 & -1 & & 0\\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & -1\\ 0 & & 1 & 1\end{vmatrix}}_{(n-1)} - (-1)\underbrace{\begin{vmatrix}1 & -1 & 0 & 0 & \cdots & 0\\ 0 & 1 & -1 & 0 & \cdots & 0\\ 0 & 1 & 1 & -1 & 0 \cdots & 0\\ & & & \ddots & \ddots & -1\\ & & & & 1 & 1\end{vmatrix}}_{(n-1)}$$

by cofactor exp. along first row

$$= 1\cdot F_{n-1} + 1\left(1\cdot\underbrace{\begin{vmatrix}1 & -1 & & 0\\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & -1\\ 0 & & 1 & 1\end{vmatrix}}_{(n-2)}\right)$$

cofactor exp. of 2nd determinant along first column

$$= F_{n-1} + F_{n-2}.$$

28) a) $C_1 = |0| = 0$

$$C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$$

$$C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1 \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1\, C_2 = 1$$

b) In general, $\boxed{C_n = -C_{n-2}}$

$$\therefore C_{10} = -C_8 = -(-C_6) = -(-(-C_4)) = -(-(-1)) = \boxed{-1}$$

34) a) <u>Claim</u>: $\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A|\,|D|$

"<u>proof:</u>" Assume A, B, C, D all $n \times n$ (so that $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is $(2n) \times (2n)$).

To find $\begin{vmatrix} A & B \\ 0 & D \end{vmatrix}$, we could do $2n$ steps of Gaussian elimination (GE) until we have an upper Δ matrix, and then we'd compute $\begin{vmatrix} A & B \\ 0 & D \end{vmatrix}$ by multiplying the diagonal entries.

<u>However, because of the zero block in the lower left corner</u>, this is the same as doing n steps of GE on A, followed by n steps of GE on D! We get the same diagonal entries either way, so

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A| \cdot |D|.$$

☑

#34, cont.

b)
$$\begin{array}{c} A \leftarrow \\ \\ \\ C \leftarrow \end{array} \left| \begin{array}{cccc} 1 & 2 & -1 & -5 \\ 6 & 3 & 2 & 6 \\ \hline 3 & 4 & 1 & 7 \\ 3 & -2 & -5 & 3 \end{array} \right| \begin{array}{c} \rightarrow B \\ \\ \\ \rightarrow D \end{array} = -868$$

but $|A||D| - |B||C| = -270 \neq -868$.

<u>Note</u>: MATLAB computes determinants!
use command: det(A).

c) Same counterexample as in (b):
$\det(AD - CB) = -1084 \neq -868$.

(36) $\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$ (assuming A^{-1} exists!)

$\Rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$

Taking determinants of both sides yields:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} I & 0 \\ CA^{-1} & I \end{vmatrix}\begin{vmatrix} A & B \\ 0 & D-CA^{-1}B \end{vmatrix}$$

$= 1 \cdot \begin{vmatrix} A & B \\ 0 & D-CA^{-1}B \end{vmatrix}$ because product of diag. entries of $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}$ is 1.

$= |A| \cdot |D - CA^{-1}B|$ by #34 (a)

For the second equality, note that if $AC = CA$

$$|A| \cdot |D - CA^{-1}B| = |A(D - CA^{-1}B)|$$
$$= |AD - \underbrace{ACA^{-1}}B|$$
$$= |AD - CB| \quad \text{since } ACA^{-1} = C \quad \boxed{\checkmark}$$

Let P_n denote $n \times n$ Pascal matrix.

(43) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 19 \end{vmatrix} = -1\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 6 & 10 \end{vmatrix} - 10\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 10 \end{vmatrix} + (20-1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}$

$$= \underbrace{-1\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 6 & 10 \end{vmatrix} - 10\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 10 \end{vmatrix} + 20\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}}_{= \det(P_4) = 1} - \underbrace{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}}_{= \det(P_3) = 1}$$

$= 1 - 1 = 0 \quad \boxed{\checkmark}$