Here are more homework solutions. If you want to know more about other homeworks, or you would like to see more details or have other questions, ask in class, review section or office hour.

Section 4.2, Problem 2

We first need to find separated solutions of the form

$$u(x,t) = X(x)T(t).$$

We proceed as in Section 4.1, pages 84/85. It follows from the formulas on top of page 85 that we get two ordinary differential equations of the form

$$T''(t) + \lambda c^2 T(t) = 0, \qquad X''(x) + \lambda X(x) = 0.$$

Setting $\lambda = \beta^2$, we get the well-known solutions

$$X(x) = C\cos\beta x + D\sin\beta x.$$

Next we have to find values for C, D and β. For this we use the boundary conditions

$$0 = u_x(0,t) = X'(0)T(t), \qquad 0 = u(\ell,t) = X(\ell)T(t).$$

We deduce from this

$$0 = X'(0) = -C\beta\sin\beta 0 + D\beta\cos\beta 0 \ = \ D\beta.$$

Hence we can assume $D = 0$ and $X(x) = C\cos\beta x$. Plugging this into the boundary condition for $x = \ell$, we get

$$0 = X(\ell) = C\cos\beta\ell.$$

It follows from this that

$$\beta\ell = (n + \frac{1}{2})\pi, \qquad n = 0, 1, 2,$$

We conclude that we get eigenfunctions of the form $\cos\frac{(n+1/2)\pi}{\ell}x$, as claimed.

To prove (b), we just solve the differential equation $T''(t) + \beta^2 c^2 T(t) = 0$ from above for $\beta = (n + \frac{1}{2})\pi/\ell$. Hence we get a separated solution

$$X(x)T(t) = (A\cos\frac{(n+1/2)\pi}{\ell}ct \ + \ B\sin\frac{(n+1/2)\pi}{\ell}ct)\cos\frac{(n+1/2)\pi}{\ell}x.$$

This leads to the series solution

$$u(x,t) \ = \ \sum_{n=0}^{\infty}(A_n\cos\frac{(n+1/2)\pi}{\ell}ct \ + \ B_n\sin\frac{(n+1/2)\pi}{\ell}ct)\cos\frac{(n+1/2)\pi}{\ell}x.$$

Section 4.3, Problem 2

If $\lambda = 0$, we have to consider the differential equation $-X''(x) = 0$. This has the easy solution

$$X(x) = ax + b.$$

It follows that $X'(x) = a$. Hence the Robin boundary condition for $x = 0$ becomes

$$0 = X'(0) - a_0 X(0) = a - a_0 b, \quad i.e. \quad a = a_0 b.$$

Substituting $a = a_0 b$ into the boundary condition for $x = \ell$, we get

$$0 = X'(\ell) + a_\ell X(\ell) = a_0 b + a_\ell (b a_0 \ell + b).$$

Dividing the right hand side of this equation by b, we obtain

$$0 = a_0 + a_\ell + a_0 a_\ell \ell.$$

This shows (a), i.e. we can only have eigenvalue $\lambda = 0$ if this identity holds. To solve (b), we observe from our work so far that any eigenfunction with eigenvalue $\lambda = 0$ satisfying the given boundary conditions must be of the form $X(x) = b(a_0 x + 1)$, as $a = a_0 b$. It is now straightforward to check that $a_0 x + 1$ does indeed satisfy both Robin boundary conditions, using (a) (check this for yourself explicitly!)

Section 4.3, Problem 4

This was done in class.

Section 3.2, Problem 1

There was a sign mistake in the posted solution, it should read $+\phi_{ev}(x - ct)$ instead of $-\phi_{ev}(x - ct)$, and also $+\phi(x - ct)$ in the subsequent formula.