VIBRATIONS OF A DRUMHEAD

These are supplementary notes to Chapter 10.2 in the book of W. Strauss with the same title. We will refer to it as just *the book*. The notes do not contain any additional material. I have only added a few more details, and I have changed the presentation a little to something I personally like better.

Let $D \subset \mathbf{R}^2$ be the disk of radius a, given by $x^2 + y^2 \leq a^2$. We consider the two-dimensional wave equation

$$u_{tt} = c^2 \Delta u = c^2 (u_{xx} + u_{yy}),$$

with the boundary condition $u = 0$ on the boundary of D, and the usual initial conditions

$$u(x, y, 0) = \phi(x, y), \qquad u_t(x, y, 0) = \psi(x, y).$$

We are going to use polar coordinates to solve this problem. Recall that

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}. \tag{1}$$

We now consider solutions of the form $u(t, r, \theta) = T(t)R(r)\Theta(\theta)$. We separate variables as usual, and obtain

$$\frac{T''}{c^2 T} = -\lambda = \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2 \Theta}. \tag{2}$$

Observe that if we set $\tilde{u}(r, \theta) = R(r)\Theta(\theta)$, the second equality of (2) together with (1) imply

$$\Delta \tilde{u} = -\lambda \tilde{u}. \tag{3}$$

If we again separate variables in the second equality of (2), we obtain

$$\frac{\Theta''}{\Theta} = -\gamma = -\lambda r^2 + \frac{r^2 R''}{R} + \frac{rR'}{R}.$$

From the first equality, we obtain that Θ is a linear combination of $\cos \sqrt{\gamma}\theta$ and $\sin \sqrt{\gamma}\theta$. The equality $\Theta(\theta + 2\pi) = \Theta$, by periodicity of θ for polar coordinates implies that $\gamma = n^2$ and

$$\Theta(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta),$$

$$T(t) = C_n \cos(\lambda ct) + D_n \sin(\lambda ct),$$

$$R'' + \frac{1}{r} R_r + (\lambda - \frac{n^2}{r^2}) R = 0,$$

with the boundary conditions $R(0)$ finite and $R(a) = 0$. The latter condition will put restrictions on the choice of λ, as we shall see. Substituting $\rho = \sqrt{\lambda} r$, one checks that this ODE transforms to Bessel's equation

$$R_{\rho\rho} + \frac{1}{\rho} R_\rho + (1 - \frac{n^2}{\rho^2}) R = 0.$$

Setting up the power series $R(\rho) = \rho^n \sum_k^\infty a_k \rho^k$, and choosing $a_0 = 1/(2^n n!)$, one obtains the solution $R = J_n$ given by (see p 266)

$$J_n(\rho) = \sum_{j=0}^{\infty} (-1)^j \frac{(\rho/2)^{2j}}{j!(n+j)!}.$$

These are called *Bessel functions*, which look like damped cosines, see the picture and discussion on p. 267 of the book. In particular, one can show that J_n has infinitely many zeros, which we can arrange as

$$0 < z_{n,1} < z_{n,2} < z_{n,3} < \ \dots\ ,$$

where we ignore the zero $z = 0$ of $J_n(\rho)$ for $n > 0$. Now the boundary condition $R(a) = 0$ implies $J_n(\sqrt{\lambda}a) = 0$. Hence $\lambda = \lambda_{n,m}$ for one of the positive numbers $\lambda_{n,m}$ determined by

$$\sqrt{\lambda_{n,m}}a = z_{n,m}.$$

In particular, this also implies that we obtain solutions

$$u^{(c,c)}_{n,m} = \cos(\sqrt{\lambda_{n,m}}ct)J_n(\sqrt{\lambda_{n,m}}r)\cos(n\theta),$$

$$u^{(c,s)}_{n,m} = \cos(\sqrt{\lambda_{n,m}}ct)J_n(\sqrt{\lambda_{n,m}}r)\sin(n\theta),$$

$$u^{(s,c)}_{n,m} = \sin(\sqrt{\lambda_{n,m}}ct)J_n(\sqrt{\lambda_{n,m}}r)\cos(n\theta),$$

$$u^{(s,s)}_{n,m} = \sin(\sqrt{\lambda_{n,m}}ct)J_n(\sqrt{\lambda_{n,m}}r)\sin(n\theta).$$

These solutions turn out to be linearly independent for $n > 0$. If $n = 0$, $u^{(c,s)}_{0,m} = 0 = u^{(s,s)}_{0,m}$. Hence we obtain

$$u^{(c,c)}_{0,m} = u^{(c)}_{n,m} = \cos(\sqrt{\lambda_{n,m}}ct)J_n(\sqrt{\lambda_{n,m}}r),$$

$$u^{(s,c)}_{0,m} = u^{(s)}_{n,m} = \sin(\sqrt{\lambda_{n,m}}ct)J_n(\sqrt{\lambda_{n,m}}r).$$

Finally, we also obtain functions $\tilde{u}^{(c)}_{n,m}$ and $\tilde{u}^{(s)}_{n,m}$ from the functions above by removing their $\cos(\sqrt{\lambda_{n,m}}ct)$ resp $\sin(\sqrt{\lambda_{n,m}}ct)$ factor. E.g. we have

$$\tilde{u}^{(c)}_{n,m} = \cos(n\theta)J_n(\sqrt{\lambda_{n,m}}r).$$

Theorem (a) The functions $\tilde{u}^{(\epsilon)}_{n,m}$, where $\epsilon = c$ or $\epsilon = s$, are eigenfunctions with eigenvalue $\lambda = \lambda_{n,m}$ of $\Delta u = -\lambda u$.

(b) All these eigenfunctions are mutually orthogonal with respect to the real inner product

$$(f, g) = \int\int_D f(x, y)g(x, y)\ dx\ dy.$$

(c) In particular, we also have

$$\int_0^a J_n(\sqrt{\lambda_{n,m}}r)J_n(\sqrt{\lambda_{n,m'}}r)r\ dr = 0 \quad \text{if } m \neq m'.$$

Proof. Statement (a) follows from (3). Now observe that by using polar coordinates we obtain

$$(\tilde{u}^{(\epsilon)}_{n,m}, \tilde{u}^{(\epsilon')}_{n',m'}) \ = \ (\int_0^{2\pi} \epsilon(n\theta)\epsilon'(n'\theta)\ d\theta)(\int_0^a J_n(\sqrt{\lambda_{n,m}}r)J_{n'}(\sqrt{\lambda_{n',m'}}r)r\ dr),$$

where ϵ and ϵ' are sin or cos (any possible combination). It was shown last quarter that the first integral is equal to 0 unless $\epsilon = \epsilon'$ and $n = n'$. Assuming this, it follows from (3) that $\tilde{u}^{(\epsilon)}_{n,m}$ and $\tilde{u}^{(\epsilon)}_{n,m'}$ are eigenfunctions of $\Delta\tilde{u} = -\lambda\tilde{u}$ with different eigenvalues $\lambda_{n,m}$ and $\lambda_{n,m'}$, unless $m = m'$. Hence the second integral is equal to 0 for $m \neq m'$, by Theorem 1 in Section 10.1. This proves (b) and (c).

In the following $L^2(D)$ denotes all measurable functions f on the disk D of radius a for which

$$\int_D f^2(x, y)\ dxdy < \infty.$$

The set of functions $\tilde{u}_{n,m}^{(\nu)}$ being an L^2-basis for $L^2(D)$ means that for any L^2 function f on D we can find coefficients $a_{n,m}^{(\nu)}$ such that

$$\int_D |f(x,y) - \sum a_{n,m}^{(\nu)} \tilde{u}_{n,m}^{(\nu)}(x,y)|^2 \, dxdy = 0.$$

This means, roughly speaking, that f is not necessarily equal to $\sum a_{n,m}^{(\nu)} \tilde{u}_{n,m}^{(\nu)}$ everywhere, but that the points on which they differ have measure zero. The following theorem uses the fact that the functions $\tilde{u}_{n,m}^{(\nu)}$ form an orthogonal L^2-basis for $L^2(D)$. This is a known result, which we will not prove in this course.

Theorem The solution of the wave equation, as stated at the beginning of this section, is given by the series

$$u(r,\theta,t) = \sum_{m,n,\epsilon,\nu} a_{n,m}^{(\epsilon,\nu)} u_{n,m}^{(\epsilon,\nu)},$$

where $n = 0, 1, 2, \; ..., \; m = 1, 2, 3, \; ...$ and ϵ, ν are sin or cos. The coefficients are calculated by

$$a_{n,m}^{(c,\nu)} = \frac{(\phi, \tilde{u}_{n,m}^{(\nu)})}{(\tilde{u}_{n,m}^{(\nu)}, \tilde{u}_{n,m}^{(\nu)})},$$

$$a_{n,m}^{(s,\nu)} = \frac{(\psi, \tilde{u}_{n,m}^{(\nu)})}{\sqrt{\lambda_{n,m}} c \; (\tilde{u}_{n,m}^{(\nu)}, \tilde{u}_{n,m}^{(\nu)})}.$$

Proof. This is proved the usual way, checking the initial conditions. Observe that for $t = 0$ we have $u_{n,m}^{(s,\nu)} = 0$, as $\sin 0 = 0$, and $u_{n,m}^{(c,\nu)} = \tilde{u}_{n,m}^{(\nu)}$, as $\cos 0 = 1$. Hence we obtain

$$\phi(r,\theta) = u(r,\theta,0) = \sum_{m,n,\nu} a_{n,m}^{(c,\nu)} \tilde{u}_{n,m}^{(\nu)}(r,\theta).$$

We now obtain the coefficient $a_{n_o,m_o}^{(c,\nu_o)}$ by taking inner products of both sides with $\tilde{u}_{n_o,m_o}^{(\nu_o)}$. We similarly obtain the coefficient $a_{n_o,m_o}^{(s,\nu_o)}$ by differentiating the series expansion for the solution u with respect to t and setting it equal to ψ for $t = 0$. By the discussion before this theorem, the ensuing power series satisfies the initial conditions except possibly for a set of measure zero, and it is a solution of the wave equation.