NOTES FOR MIKE FREEDMAN

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1. The Temperley-Lieb Category

1.1. **The Generic Temperley Lieb Category.** Let t be an indeterminant over $\mathbb{C}$, and let $d = (t + t^{-1})$. The *generic Temperley Lieb category* TL is a strict tensor categor whose *objects* are elements of $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. The set of *morphisms* $\mathrm{Hom}(m, n)$ from m to n is a $\mathbb{C}(t)$ vector space described as follows:

If $n - m$ is odd, then $\mathrm{Hom}(m, n)$ is the zero vector space.

For $n - m$ even, we first define (m, n)–TL diagrams, consisting of:

1. a closed rectangle R in the plane with two opposite edges designated as top and bottom.
2. m marked points (vertices) on the top edge and n marked points on the bottom edges.
3. $(n + m)/2$ smooth curves (or "strands") in R such that for each curve γ, $\partial\gamma = \gamma \cap \partial R$ consists of two of the $n + m$ marked points, and such that the curves are pairwise non-intersecting.

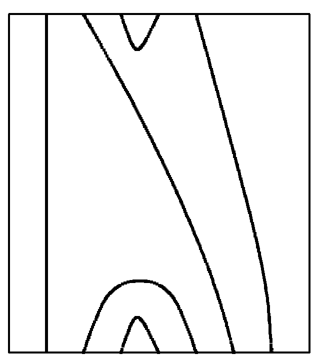


Figure 1. A (5,7)–Temperley Lieb Diagram

Two such diagrams are *equivalent* if they induce the same pairing of the $n+m$ marked points. $\mathrm{Hom}(m, n)$ is defined to be the $\mathbb{C}(t)$ vector space with basis the set of equivalence classes of (n, m)–TL diagrams; we will refer to equivalence classes of diagrams simply as diagrams.

The composition of morphisms is defined first on the level of diagrams. The composition ba of an (m, n)–diagram b and an (ℓ, m)–diagram a is defined by the following steps:

1. Juxtapose the rectangles of a and b, identifying the bottom edge of a (with its m marked points) with the top edge of b (with its m marked points).
2. Remove from the resulting rectangle any closed loops in its interior. The result is a (n, ℓ)–diagram c.
3. The product ba is $d^r c$, where r is the number of closed loops removed.

The composition product evidently respects equivalence of diagrams, and extends uniquely to a bilinear product

$$\mathrm{Hom}(m, n) \times \mathrm{Hom}(\ell, m) \longrightarrow \mathrm{Hom}(\ell, n),$$

hence to a linear map

$$\mathrm{Hom}(m, n) \otimes \mathrm{Hom}(\ell, m) \longrightarrow \mathrm{Hom}(\ell, n).$$

The *tensor product of objects* in TL is given by $n \otimes n' = n + n'$. The *tensor product of morphisms* is defined by horizontal juxtposition. More exactly, The tensor $a \otimes b$ of an (n, m)–TL diagram a and an (n', m')–diagram b is defined by horizontal juxtposition of the diagrams, the result being an $(n + n', m + m')$–TL diagram.

The tensor product extends uniquely to a bilinear product

$$\mathrm{Hom}(m, n) \times \mathrm{Hom}(m', n') \longrightarrow \mathrm{Hom}(m + m', n + n'),$$

hence to a linear map

$$\mathrm{Hom}(m, n) \otimes \mathrm{Hom}(m', n') \longrightarrow \mathrm{Hom}(m + m', n + n').$$

For each $n \in \mathbb{N}_0$, $T_n := \mathrm{End}(n)$ is a $\mathbb{C}(t)$–algebra, with the composition product. The identity 1_n of $T(n)$ is the diagram with n vertical (non-crossing) strands. We have canonical embeddings of T_n into T_{n+m} given by $x \mapsto x \otimes 1_m$. If $m > n$ with $m - n$ even, there also exist obvious embeddings of $\mathrm{Hom}(n, m)$ and $\mathrm{Hom}(m, n)$ into T_m as follows: If $\cap$ and $\cup$ denote the morphisms in $\mathrm{Hom}(2, 0)$ and $\mathrm{Hom}(0, 2)$, then we have linear embeddings

$$a \in \mathrm{Hom}(n, m) \mapsto a \otimes \cup^{\otimes (m-n)/2} \in T_m$$

and

$$b \in \mathrm{Hom}(m, n) \mapsto b \otimes \cap^{\otimes (m-n)/2} \in T_m.$$

Note that these maps have left inverses which are given by premultiplication by an element of $\mathrm{Hom}(n, m)$ in the first case, and postmultiplication by an element of $\mathrm{Hom}(m, n)$ in the second. Namely,

$$a = d^{-(m-n)/2} (a \otimes \cup^{\otimes (m-n)/2}) \circ (\mathbf{1}_n \otimes \cap^{\otimes (m-n)/2})$$

and

$$b = d^{-(m-n)/2} (\mathbf{1}_n \otimes \cup^{\otimes (m-n)/2}) \circ (b \otimes \cap^{\otimes (m-n)/2})$$

By an *ideal* J in TL we shall mean a vector subspace of $\bigoplus_{n,m} \mathrm{Hom}(n, m)$ which is closed under composition and tensor product with arbitrary morphisms. That is, if a, b are composible morphisms, and one of them is in J, then the composition ab is in J; and if a, b are any morphisms, and one of them is in J, then the tensor product $a \otimes b$ is in J.

Note that any ideal is closed under the embeddings described just above, and under their left inverses.

1.2. **Specializations and evaluable morphisms.** For any $\tau \in \mathbb{C}$, we define the specialization $\mathrm{TL}(\tau)$ of the Temperley Lieb category at τ, which is obtained by replacing the indeterminant t by τ. More exactly, the objects of $\mathrm{TL}(\tau)$ are again elements of $\mathbb{N}_0$, the set of morphisms $\mathrm{Hom}(m,n)(\tau)$ is the $\mathbb{C}$–vector space with basis the set of (m,n)–TL diagrams, and the composition rule is as before, except that d is replaced by $d(\tau) = (\tau + \tau^{-1})$. Tensor products are defined as before. $T_n(\tau) := \mathrm{End}(n)$ is a complex algebra, and $x \mapsto x \otimes 1_m$ defines a canonical embedding of $T_n(\tau)$ into $T_{n+m}(\tau)$. One also has embeddings $\mathrm{Hom}(m,n) \to T_n$ and $\mathrm{Hom}(n,m) \to T_n$, when $m < n$, as before. An ideal in $\mathrm{TL}(\tau)$ again means a subspace of $\bigoplus_{n,m} \mathrm{Hom}(n,m)$ which is closed under composition and tensor product with arbitrary morphisms.

Let $\mathbb{C}(t)_\tau$ be the ring of rational functions without pole at τ. The set of *evaluable* morphisms in $\mathrm{Hom}(m,n)$ is the $\mathbb{C}(t)_\tau$–span of the basis of (n,m)–TL diagrams. Note that the composition and tensor product of evaluable morphisms are evaluable. We have an *evaluation map* from the set of evaluable morphisms to morphisms of $\mathrm{TL}(\tau)$ defined by

$$a = \sum s_j(t) a_j \mapsto a(\tau) = \sum s_j(\tau) a_j,$$

where the s_j are in $\mathbb{C}(t)_\tau$, and the a_j are TL-diagrams. We write $x \mapsto x(\tau)$ for the evaluation map. The evaluation map is a homomorphism for the composition and tensor products. In particular, one has a $\mathbb{C}$–algebra homomorphism from the algebra T_n^τ of evaluable endomorphisms of n to the algebra $T_n(\tau)$ of endomorphisms of n in $\mathrm{TL}(\tau)$.

The principle of constancy of dimension is an important tool for analyzing the specialized categories $\mathrm{TL}(\tau)$. We state it in the form which we need here:

Proposition 1.1. *Let $e \in T_n$ and $f \in T_m$ be evaluable idempotents in the generic Temperley Lieb category. Let A be the $\mathbb{C}(t)$–span in $\mathrm{Hom}(m,n)$ of a certain set of (m,n)–TL diagrams, and let $A(\tau)$ be the $\mathbb{C}$–span in $\mathrm{Hom}(m,n)(\tau)$ of the same set of diagrams. Then*

$$\dim_{\mathbb{C}(t)} eAf = \dim_{\mathbb{C}} e(\tau)A(\tau)f(\tau).$$

Proof. Let X denote the set of TL diagrams spanning A. Clearly

$$\dim_{\mathbb{C}(t)} A = \dim_{\mathbb{C}} A(\tau) = |X|.$$

Choose a basis of $e(\tau)A(\tau)f(\tau)$ of the form $\{e(\tau)xf(\tau) : x \in X_0\}$, where X_0 is some subset of X. If the set $\{exf : x \in X_0\}$, were linearly dependent over $\mathbb{C}(t)$, then it would be linearly dependent over $\mathbb{C}[t]$, and evaluating at τ would give a linear dependence of $\{e(\tau)xf(\tau) : x \in X_0\}$ over $\mathbb{C}$. It follows that

$$\dim_{\mathbb{C}(t)} eAf \geq \dim_{\mathbb{C}} e(\tau)A(\tau)f(\tau).$$

But one has similar inequalities with e replaced by $\mathbf{1} - e$ and/or f replaced by $\mathbf{1} - f$. If any of the inequalities were strict, then adding them would give $\dim_{\mathbb{C}(t)} A > \dim_{\mathbb{C}} A(\tau)$, a contradiction. □

1.3. **The Markov trace.** The Markov trace $\mathrm{Tr} = \mathrm{Tr}_n$ is defined on T_n (or on $T_n(\tau)$) by the following picture, which represents an element in $\mathrm{End}_0 \cong \mathbb{C}(t)$ (resp. $\mathrm{End}(0) \cong \mathbb{C}$).

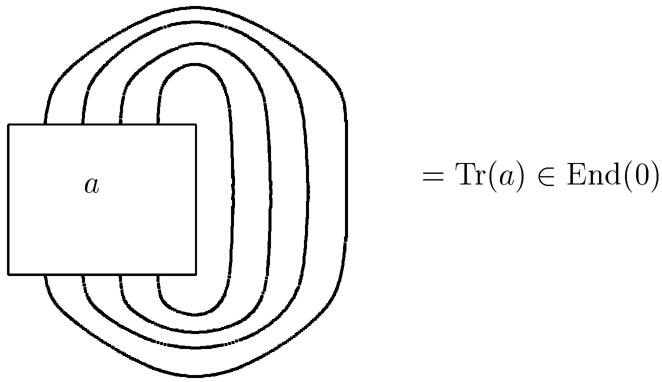


FIGURE 2. The categorical trace of an element $a \in T_n$.

On an (n,n)–TL diagram $a \in T_n$, the trace is evaluated geometrically by closing up the diagram as in the figure, and counting the number $c(a)$ of components (closed loops); then $\mathrm{Tr}(a) = d^{c(a)}$.

It will be useful to give the following inductive description of closing up a diagram. We define a map $\varepsilon_n : T_{n+1} \to T_n$ (known as a conditional expectation in operator algebras) by only closing up the last strand; algebraically it can be defined by

$$a \in T_{n+1} \quad \mapsto \quad (1_n \otimes \cup) \circ (a \otimes 1) \circ (1_n \otimes \cap).$$

If $k > n$, the map $\varepsilon_{n,k}$ is defined by $\varepsilon_{n,k} = \varepsilon_n \circ \varepsilon_{n+1} \ ... \ \circ \varepsilon_{k-1}$. It follows from the definitions that $\mathrm{Tr}(a) = \varepsilon_{0,n}$ for $a \in T_n$.

It is well-known that Tr is indeed a functional satisfying $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$; one easily checks that this equality is even true if $a \in \mathrm{Hom}(n,m)$ and $b \in \mathrm{Hom}(m,n)$. We need the following well-known fact:

Lemma 1.2. *Let $f \in T_{n+m}$ and let $p \in T_n$ such that $(p \otimes 1_m) f (p \otimes 1_m) = f$, where p is a minimal idempotent in T_n. Then $\varepsilon_{n,n+m}(f) = \gamma p$, where $\gamma = Tr_{n+m}(f)/Tr_n(p)$*

Proof. It follows from the definitions that

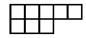
$$p\varepsilon_{n,n+m}(f)p = \varepsilon_{n,n+m}((p \otimes 1_m) f (p \otimes 1_m)) = \varepsilon_{n,n+m}(f).$$

As p is a minimal idempotent in T_n, $\varepsilon_{n,n+m}(f) = \gamma p$, for some scalar γ. Moreover, by our definition of trace, we have $Tr_{n+m}(f) = Tr_n(\varepsilon_{n,n+m}(f)) = \gamma Tr_n(p)$. This determines the value of γ. □

The negligible morphisms $\mathrm{Neg}(n,m)$ are defined to be all elements $a \in \mathrm{Hom}(n,m)$ for which $Tr(ab) = 0$ for all $b \in \mathrm{Hom}(m,n)$. It is well-known that the negligible morphisms form an ideal in TL.

2. The structure of the Temperley Lieb algebras

2.1. **The generic Temperley Lieb algebras.** Recall that a *Young diagram* $\lambda = [\lambda_1, \lambda_2, \ \dots \ \lambda_k]$ is a left justified array of boxes with λ_i boxes in the i-th row and $\lambda_i \geq \lambda_{i+1}$ for all i. For example,

$[5, 3] =$  .

All Young diagrams in this paper will have at most two rows. For λ a Young diagram with n boxes, a *Young tableau of shape* λ is a filling of λ with the numbers 1 through n so that the numbers increase in each row and column. The number of Young tableax of shape λ is denoted by f_λ.

The generic Temperley Lieb algebras T_n are known ([J]) to decompose as direct sums of full matrix algebras over the field $\mathbb{C}(t)$, $T_n = \bigoplus_\lambda T_\lambda$, where the sum is over all Young diagrams λ with n boxes (and with no more than two rows), and T_λ is isomorphic to an f_λ-by-f_λ matrix algebra.

When λ and μ are Young diagrams of size n and $n+1$, one has a (non-unital) homomorphism of T_λ into T_μ given by $x \mapsto (x \otimes 1) z_\mu$, where z_μ denotes the minimal central idempotent in T_{n+1} such that $T_\mu = T_{n+1} z_\mu$. Let $g_{\lambda,\mu}$ denote the rank of $(e \otimes 1) z_\mu$, where e is any minimal idempotent in T_λ. It is known that $g_{\lambda,\mu} = 1$ in case μ is obtained from λ by adding one box, and $g_{\lambda,\mu} = 0$ otherwise.

One can describe the embedding of T_n into T_{n+1} by a *Bratteli diagram* (or induction-restriction diagram), which is a bipartite graph with vertices labelled by two-row Young diagrams of size n and $n+1$ (corresponding to the simple components of T_n and T_{n+1}) and with $g_{\lambda,\mu}$ edges joining the vertices labelled by λ and μ. That is λ and μ are joined by an edge precisely when μ is obtained from λ by adding one box. The sequence of embeddings $T_0 \to T_1 \to T_2 \to \dots$ is described by a multilevel Bratteli diagram, as shown in Figure 3.

A tableau of shape λ may be identified with an increasing sequence of Young diagrams beginning with the empty diagram and ending at λ; namely the j-th diagram in the sequence is the subdiagram of λ containing the numbers 1, 2, ... , j. Such a sequence may also be interpreted as a *path* on the Bratteli diagram of Figure 3, beginning at the empty diagram and ending at λ.

2.2. **Path idempotents.** One can define a familiy of minimal idempotents p_t in T_n, labelled by paths t of length n on the Bratteli diagram (or equivalently, by Young tableaux of size n), with the following properties:

1. $p_t p_s = 0$ if t, s are different paths both of length n.
2. $z_\lambda = \sum \{p_t : t \text{ ends at } \lambda\}$.
3. $p_t \otimes 1 = \sum \{p_s : s \text{ has length } n+1 \text{ and extends } t\}$

Let t be a path of length n and shape λ and let μ be a Young diagram of size $n+m$. It follows that $(p_t \otimes 1_m) z_\mu \neq 0$ precisely when there is a path on the Bratteli diagram from λ to μ. It has been shown in [J] that (in our notations) $\mathrm{Tr}(p_t) = [\lambda_1 - \lambda_2 + 1]$, where $[m] = (t^m - t^{-m})/(t - t^{-1})$ for any

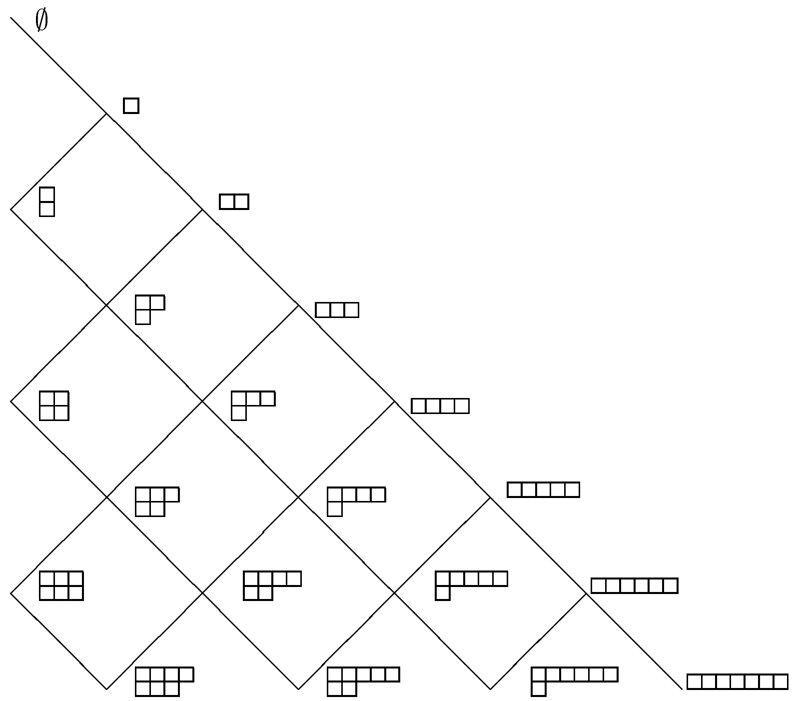


FIGURE 3. Bratteli diagram for the sequence (T_n)

integer m, and where λ is the endpoint of the path t. Observe that we get the same value for diagrams λ and μ which are in the same column in the Bratteli diagram.

The idempotents p_t were defined by recursive formulas in [W2], generalizing the formulas for the Jones-Wenzl idempotents in [W1].

2.3. **Specializations at non-roots of unity.** When τ is not a proper root of unity, the Temperley Lieb algebras $T_n(\tau)$ are semi-simple complex algebras with the "same" structure as generic Temperley Lieb algebras. That is, $T_n(\tau) = \bigoplus_\lambda T_\lambda(\tau)$, where $T_\lambda(\tau)$ is isomorphic to an f_λ-by-f_λ matrix algebra over $\mathbb{C}$. The embeddings $T_n(\tau) \to T_{n+1}(\tau)$ are described by the Bratteli diagram as before. The idempotents p_t, and the minimal central idempotents z_λ, in the generic algebras T_n, are evaluable at τ, and the evaluations $p_t(\tau)$, resp. $z_\lambda(\tau)$, satisfy analogous properties.

2.4. **Specializations at roots of unity and evaluable idempotents.** We require some terminology for discussing the case where τ is a root of unity. Let $q = \tau^2$, and suppose that q is a primitive ℓ-th root of unity. We say that a Young diagram λ is *critical* if $w(\lambda) := \lambda_1 - \lambda_2 + 1$ is divisible by ℓ. The m-th *critical line* on the Bratteli diagram for the generic Temperly Lieb algebra is the line containing the diagrams λ with $w(\lambda) = ml$. See Figure 4.

Say that two non-critical diagrams λ and μ with the same number of boxes are *reflections of one another in the m-th critical line* if $\lambda \neq \mu$ and $|w(\lambda) - m\ell| = |w(\mu) - m\ell| < \ell$. (For example, with $\ell = 3$, $[2,2]$ and $[4]$ are reflections in the first critical line $w(\lambda) = 3$.)

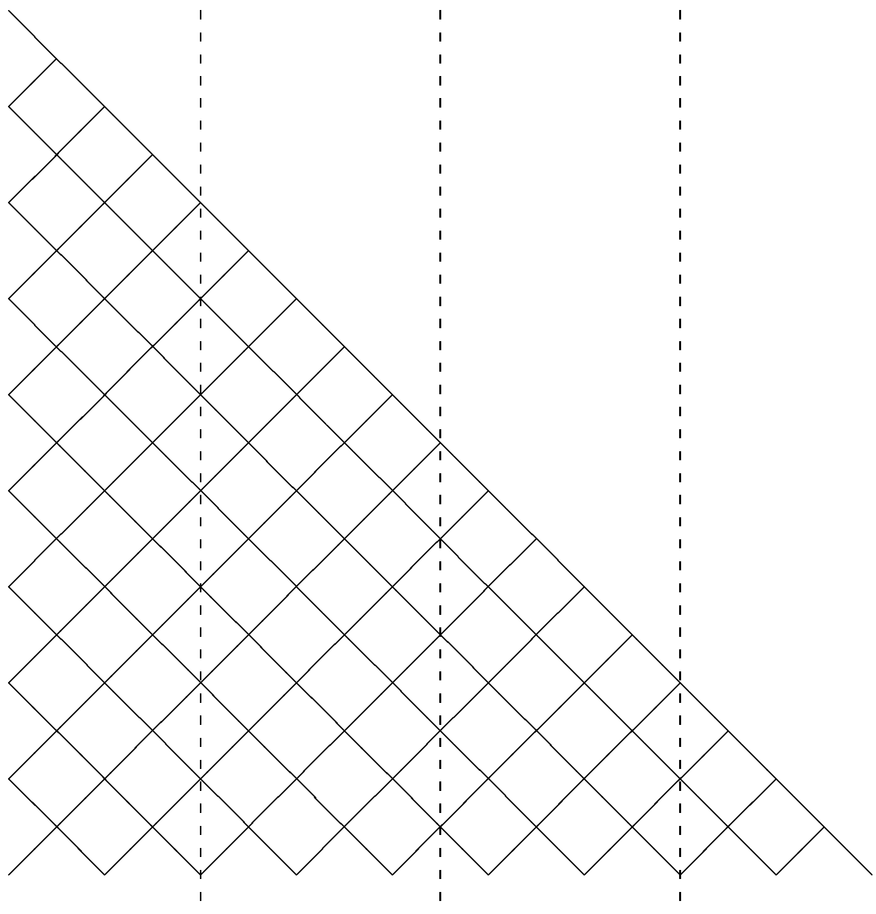


FIGURE 4. Critical lines

For τ a proper root of unity, the formulas for path idempotents in [W1] and [W2] generally contain poles at τ, i.e. the idempotents are not evaluable. However, suitable sums of path idempotents are evaluable.

Suppose $w(\lambda) \leq \ell$ and t is a path of shape λ which stays strictly to the left of the first critical line (in case $w(\lambda) < \ell$), or hits the first critical line for the first time at λ (in case $w(\lambda) = \ell$); then p_t is evaluable at τ, and furthermore $\mathrm{Tr}(p_t) = [w(\lambda)]_\tau = (\tau^{w(\lambda)} - \tau^{-w(\lambda)})/(\tau - \tau^{-1})$.

For each critical diagram λ of size n, the minimal central idempotent z_λ in T_n is evaluable at τ. Furthermore, for each non-critical diagram λ of size n, an evaluable idempotent $z_\lambda^L = \sum p_t \in T_n$ was defined in [GW] as follows:

The summation goes over all paths t ending in λ for which the last critical line hit by t is the one nearest to λ to the left *and* over the paths obtained from such t by reflecting its part after the last critical line in the critical line (see Figure 5).

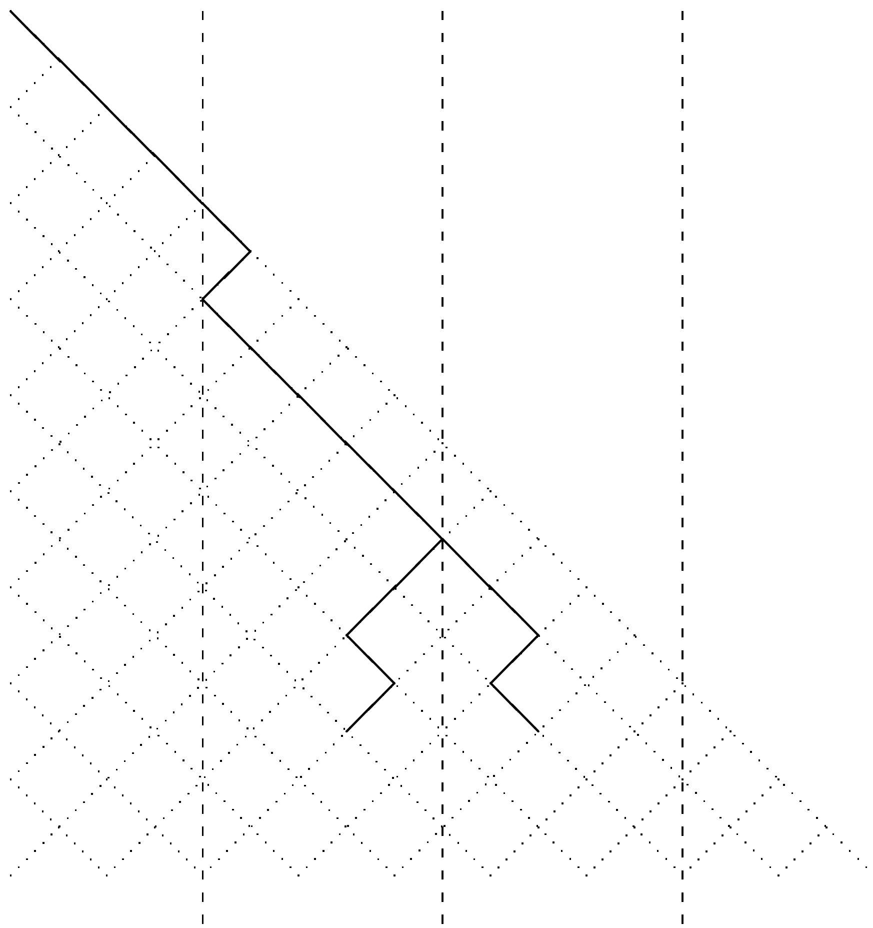


FIGURE 5. A path and its reflected path.

These idempotents have the following properties (which were shown in [GW]):

1. $\{z_\lambda(\tau) : \lambda \text{ critical }\} \cup \{z^L_\mu(\tau) : \mu \text{ non-critical }\}$ is a partition of unity in $T_n(\tau)$; that is, the idempotents are mutually orthogonal and sum to the identity.
2. $z_\lambda(\tau)$ is a minimal central idempotent in $T_n(\tau)$ if λ is critical, and $z^L_\lambda(\tau)$ is minimal central modulo the nilradical of T_n if λ is not critical (see [GW], Theorem 2.2 and Theorem 2.3).
3. For λ and μ non-critical, $z^L_\lambda(\tau)T_n(\tau)z^L_\mu(\tau) \neq 0$ only if $\lambda = \mu$, or if there is exactly one critical line between λ and μ which reflects λ to

μ. If in this case μ is to the left of λ, $z_\lambda^L T_n z_{\lambda'}^L \subseteq T_\mu$ (in the generic Temperley Lieb algebra).

4. Let $z_n^{reg} = \sum p_t$, where the summation goes over all paths t which stay strictly to the left of the first critical line, and let $z_n^{nil} = \mathbf{1} - z_n^{reg}$. Then both z_n^{reg} and z_n^{nil} are evaluable; this is a direct consequence of the fact that $z_n^{reg} = \sum_\lambda z_\lambda^L$, where the summation goes over diagrams λ with n boxes with width $w(\lambda) < \ell$.

Proposition 2.1. *The ideal of negligible morphisms in* $\mathrm{TL}(\tau)$ *is generated by the idempotent* $p_{[\ell-1]}(\tau) \in T_{\ell-1}(\tau)$.

Proof. Let us first show that $z_n^{nil}(\tau)$ is in the ideal generated by $p_{[\ell-1]}(\tau)$ for all n. This is clear for $n < \ell$, as $z_{\ell-1}^{nil} = p_{[\ell-1]}$ and $z_n^{nil} = 0$ for $n < \ell - 1$.

Moreover, z_n^{nil} is a central idempotent in the maximum semisimple quotient of T_n, whose minimal central idempotents are the z_λ^L with $w(\lambda) \geq \ell$. One checks pictorially that $p_{[\ell-1]} z_\lambda^L \neq 0$ for any such λ (i.e. the path to $[\ell-1]$ can be extended to a path t for which p_t is a summand of z_λ^L). This proves our assertion in the maximum semisimple quotient of T_n; it is well-known that in this case also the idempotent itself must be in the ideal generated by $p_{[\ell-1]}$. In particular, $\mathrm{Hom}(n,m) z_m^{nil}(\tau) + z_n^{nil}(\tau)\mathrm{Hom}(n,m)$ is also contained in this ideal.

By [GW], Theorem 2.2 (c), for λ a Young diagram of size n, with $w(\lambda) < \ell$, $z_\lambda^L T_n z_\lambda^L(\tau)$ is a full matrix algebra, which moreover contains a minimal idempotent p_t of trace $\mathrm{Tr}(p_t) = [w(\lambda)]_\tau \neq 0$. Therefore

$$z_\lambda^L T_n z_\lambda^L(\tau) \cap \mathrm{Neg}(n,n) = (0).$$

Furthermore, $z_n^{reg} T_n z_n^{reg}(\tau) = \sum z_\lambda^L T_n z_\lambda^L(\tau)$, by Fact 4 above, so

$$z_n^{reg} T_n z_n^{reg}(\tau) \cap \mathrm{Neg}(n,n) = (0)$$

as well. Now for $x \in \mathrm{Neg}(n,n)$, one has $z_n^{reg}(\tau) x z_n^{reg}(\tau) = 0$, so

$$x \in T_n(\tau) z_n^{nil}(\tau) + z_n^{nil}(\tau) T_n(\tau).$$

We have shown that $\mathrm{Neg}(n,n)$ is contained in the ideal of $\mathrm{TL}(\tau)$ generated by $p_{[l-1]}$, for all n. That the same is true for $\mathrm{Neg}(m,n)$ with $n \neq m$ follows from using the embeddings, and their left inverses, described at the end of Section 1.1.

□

3. Ideals

Proposition 3.1. *Any proper ideal in* TL *(or in* $\mathrm{TL}(\tau)$*) is contained in the ideal of negligible morphisms.*

Proof. Let $a \in \mathrm{Hom}(m,n)$. For all $b \in \mathrm{Hom}(n,m)$, $\mathrm{tr}(ba)$ is in the intersection of the ideal generated by a with the scalars $\mathrm{End}(0)$. If a is not negligible, then the ideal generated by a contains an non-zero scalar, and therefore contains all morphisms. □

Corollary 3.2. *The categories* TL *and* TL(τ) *for* τ *not a proper root of unity have no non-zero proper ideals.*

Proof. There are no non-zero negligible morphisms in TL and in TL(τ) for τ not a proper root of unity. □

Theorem 3.3. *Suppose that* τ *is a proper root of unity. Then the negligible morphisms form the unique non-zero proper ideal in* TL(τ).

Proof. Let J be a non-zero proper ideal in TL(τ). By the embeddings discussed at the end of Section 1.1, we can assume $J \cap T_n \neq 0$ for some n.

Now let a be a non-zero element of $J \cap T_n(\tau)$. Since $\{z_\lambda(\tau)\} \cup \{z_\mu^L(\tau)\}$ is a partition of unity in $T_n(\tau)$, one of the following conditions hold:

(a) $b = az_\mu(\tau) \neq 0$ for some critical diagram μ.
(b) $b = z_\mu^L(\tau) a z_\mu^L(\tau) \neq 0$ for some non-critical diagram μ.
(c) $b = z_\lambda^L(\tau) a z_{\lambda'}^L(\tau) \neq 0$ for some pair λ, λ' of non-critical diagrams which are reflections of one another in a critical line. In this case, let μ denote the leftmost of the two diagrams λ, λ'.

In each of the three cases, one has $b \in e(\tau)T_n(\tau)f(\tau)$, where e, f are evaluable idempotents in T_n such that $eT_nf \subseteq T_\mu$. Let α be a Young diagram on the first critical line of size $n+m$, such that there exists a path on the generic Bratteli diagram connecting μ and α. Then one has

$$\begin{aligned}\dim_{\mathbb{C}}\ & z_\alpha(\tau)(e(\tau)\otimes 1_m)(T_n(\tau)\otimes \mathbb{C}1_m)(f(\tau)\otimes 1_m)\\ &= \dim_{\mathbb{C}(t)} z_\alpha(e\otimes \mathrm{id}_m)(T_n\otimes \mathbb{C}(t)1_m)(f\otimes 1_m)\\ &= \dim_{\mathbb{C}(t)} eT_nf = \dim_{\mathbb{C}} e(\tau)T_n(\tau)f(\tau)\end{aligned}$$

where the first and last equalities result from the principle of constancy of dimension, and the second equality is because $x \mapsto z_\alpha(x\otimes 1_m)$ is injective from T_μ to T_α. But then it follows that $x \mapsto z_\alpha(\tau)(x\otimes 1_m)$ is injective on $e(\tau)T_n(\tau)f(\tau)$. In particular $(b\otimes 1_m)z_\alpha$ is a non-zero element of $J\cap T_\alpha$. Hence there exists $c \in T_\alpha$ such that $f = c(b\otimes 1_m)z_\alpha$ is an idempotent. After conjugating (and multiplying with $p_{[\ell-1]}\otimes 1_m$, if necessary), we can assume f to be a subidempotent of $p_{[\ell-1]}\otimes 1_m$. But then $\varepsilon_{\ell-1+m,\ell-1}(f)$ is a multiple of $p_{[\ell-1]}$, by Lemma 1.2, with the multiple equal to the rank of f in T_α. This, together with Prop. 2.1, finishes the proof. □

It is easily seen that TL has a subcategory TL^{ev} whose objects consist of even numbers of points, and with the same morphisms between sets of even points as for TL. The evaluation TL$^{ev}(\tau)$ is defined in complete analogy to TL(τ).

Corollary 3.4. *If* τ^2 *is a proper root of unity of degree* ℓ *with* ℓ *odd, the negligible morphisms form the unique non-zero proper ideal in* TL^{ev}.

Proof. If ℓ is odd, $p_{[\ell-1]}$ is a morphism in TL^{ev}. The proof of the last theorem goes through word for work (one only needs to make sure that one stays within TL^{ev}, which is easy to check). □

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