MIDTERM 2 MATH 103A Winter 2019

1. Let $\alpha = (12345)(14)$. Write α^{1802} as a product of disjoint cycles.

We first have to write α as a product of *disjoint* cycles. We obtain $\alpha = (15)(234)$. It follows that $ord(\alpha) = 6$ and hence

$$\alpha^{1802} = \alpha^{1800}\alpha^2 = (15)^2(234)^2 = (243).$$

2. How many elements of order 10 are in S_7?

The only way for an element $\alpha \in S_7$ to have order 10 is if it is a disjoint product of a 5-cycle and a two-cycle. Observe that after we have fixed the 5-cycle, there is only one choice for the remaining 2-cycle, using the remaining two numbers. So it suffices to determine how many five cycles we have in S_7.

For a given subset of five elements of $\{1, 2, \ ..., 7\}$, we have 4! different 5-cycles moving these five numbers (You can freely choose which number you write first - you then have 4! ways how to write the remaining four numbers in the cycle). Hence the number of 5-cycles is given by

$$\binom{7}{5} \ 4! \ = \ 21 \cdot 24 \ = \ 501.$$

3. Are the groups $U(5)$ and $U(10)$ isomorphic? Prove or disprove it.

We first observe that both groups have order 4. We check that $U(5)$ is cyclic, with generator 2: $2^2 = 4$, $2^3 = 3$ mod 5 and $2^4 = 1$ mod 5. We similarly check that 3 generates $U(10)$. But any cyclic group of order 4 must be isomorphic to $\mathbb{Z}_4$. Hence $U(5) \cong U(10)$. We could also explicitly check that the map $2 \in U(5) \mapsto 3 \in U(10)$ defines an isomorphism.

4. Write down the cosets of the subgroup $\{1, 9\}$ in $U(20)$.

These are $\{1, 9\}$, $\{3, 7\} = 3\{1, 9\}$, $\{11, 19\} = 11\{1, 9\}$ and $\{13, 17\} = 13\{1, 9\}$.

5. Let α be an automorphism of $\mathbb{Z}$.
(a) Is it possible that $\alpha(1) = 2$? Either prove that there is an automorphism with this property or give a reason why it can not exist.
(b) Determine all possible automorphisms of $\mathbb{Z}$.

If $\alpha(1) = 2$, then $\alpha(j) = 2j$ is even for all $j \in \mathbb{Z}$. Hence the image of α only contains even numbers. It is not surjective, and hence not an automorphism.

To prove (b), we observe that if $\alpha(1) = k$, then α is completely determined as $\alpha(j) = kj$, and hence $\alpha(\mathbb{Z}) = k\mathbb{Z}$. So we see as in (a) that α can only be surjective if $k \in \{\pm 1\}$. Obviously, the maps $\alpha(j) = j$ and $\alpha(j) = -j$ are injective and they are homomorphisms. So we have exactly two automorphisms of $\mathbb{Z}$.