

Math 102
 Winter '08
 Homework #10

6.1

④ a) $F(x,y) = -1 + 4(e^x - x) - 5x \sin y + 6y^2$ @ $(x,y) = (0,0)$

$$\frac{dF}{dx} = 4e^x - 4 - 5 \sin y$$

$$\frac{d^2F}{dx^2} = 4e^x \Big|_{(0,0)} = 4$$

$$\frac{dF}{dy} = -5x \cos y + 12y$$

$$\frac{d^2F}{dy^2} = 5x \sin y + 12 \Big|_{(0,0)} = 12$$

$$\frac{d^2F}{dy dx} = -5 \cos y \Big|_{(0,0)} = -5$$

$$\therefore F_{xx}(0,0) \cdot F_{yy}(0,0) - (F_{yx}(0,0))^2 = 4(12) - 25 > 0$$

Since $F_{xx}(0,0)$ also > 0 , $F(0,0)$ is a local min

b) $F(x,y) = (x^2 - 2x) \cos y = x^2 \cos y - 2x \cos y$ @ $(1, \pi)$

$$\frac{dF}{dx} = 2x \cos y - 2 \cos y$$

$$\frac{d^2F}{dx^2} = 2 \cos y \Big|_{(1, \pi)} = 2 \cos \pi = -2$$

$$\frac{dF}{dy} = -x^2 \sin y + 2x \sin y$$

$$\frac{d^2F}{dy^2} = -x^2 \cos y + 2x \cos y \Big|_{(1, \pi)} = -\cos \pi + 2 \cos \pi = 1 - 2 = -1$$

$$\frac{d^2F}{dy dx} = -2x \sin y + 2 \sin y \Big|_{(1, \pi)} = -2 \sin \pi + 2 \sin \pi = 0$$

$$-2(-1) - (0)^2 = 2 > 0, \quad \frac{d^2F}{dx^2} < 0 \Rightarrow F(1, \pi) \text{ is}$$

local max

6.2

$$\textcircled{2} A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \quad |2| = 2 > 0$$

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 + 1 > 0$$

$$\begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} = - \begin{vmatrix} -1 & -1 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & -1 \end{vmatrix} + 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$
$$= -(1 - (-2)) + (-2 - 1) + 2(5)$$
$$= -3 - 3 + 10 > 0$$

$\therefore A$ is P.D. by UB (III).

$$B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \quad |2| = 2 > 0$$

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 5 > 0$$

$$|B| = 4 > 0$$

$\therefore B$ is P.D.

$$C = \begin{pmatrix} 5 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix} \quad |5| = 5 > 0$$

$$\begin{vmatrix} 5 & 2 \\ 2 & 2 \end{vmatrix} > 0$$

$$|C| = 16 > 0$$

$\therefore C$ is P.D.

$\textcircled{4}$ Eigenvalues of A^2 are squares of eigenvalues of A .

Eigenvalues of A^{-1} are reciprocals of e'vals of A .

$\therefore A$ has all pos. e'vals $\Rightarrow A^2, A^{-1}$ have all pos e'vals

ie. A pos. def $\Rightarrow A^2, A^{-1}$ pos def.

⑧ A spd, C n.s.

Claim: $B = C^T A C$ is s.p.d.

(i) Show B symmetric:

$$B^T = (C^T A C)^T = C^T A^T (C^T)^T = C^T \underbrace{A^T}_A C$$

but A sym, so $A^T = A$

$$\therefore B^T = C^T A C = B$$

i.e. B is symmetric ✓

(ii) Show B is pos def.

Let $x \in \mathbb{R}^n$. Then $x^T B x = x^T C^T A C x$

$$= (C x)^T A (C x)$$

> 0 since A pos def.

Since x was arbitrary, we have that

$$x^T B x > 0 \quad \forall x \in \mathbb{R}^n$$

$\therefore B$ is pos def. ✓

(i) & (ii) $\Rightarrow B$ is s.p.d. □

③① $A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$

a) $|A| = \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} = 10$.

b) $\lambda(A) = \{2, 5\}$

c) $A \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$

$$\Rightarrow \left(A \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad A \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \right) = \left(2 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad 5 \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \right)$$

↑ ↑
eigenvectors of A .

30, cont.

6.2

d) A is pos. def b/c $\lambda > 0 \forall \lambda \in \Lambda(A)$.

A is symmetric since

$$A^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^T \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}^T \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^T$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A \checkmark$$

34) $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

pivot > 1 pivot > 1

∴ all pivots of A are > 1, but $\lambda = .5858$ is an e-value of A.

SO the answer is **NO**.

6.3

2) $AA^T = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 17 & 34 \\ 34 & 68 \end{pmatrix}$

$\det(AA^T - \lambda I) = 0 \Leftrightarrow \begin{vmatrix} 17-\lambda & 34 \\ 34 & 68-\lambda \end{vmatrix} = 0 \Leftrightarrow (17-\lambda)(68-\lambda) - 1156 = 0$

$\Leftrightarrow \lambda = 0, 85$.

$\lambda = 0$
 σ_2^2 $\begin{pmatrix} 17 & 34 & | & 0 \\ 34 & 68 & | & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 17 & 34 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \Rightarrow x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$

∴ $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is evec CORR TO $\lambda = 0$. $\| \begin{pmatrix} -2 \\ 1 \end{pmatrix} \| = \sqrt{4+1} = \sqrt{5}$

∴ $u_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$.

$\lambda = 85$
 σ_1^2 $\begin{pmatrix} -68 & 34 & | & 0 \\ 34 & -17 & | & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -68 & 34 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \Rightarrow -2x_1 + x_2 = 0$
 $\Rightarrow x_2 = 2x_1$

∴ $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is evec CORR TO $\lambda = 85$.

∴ $u_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$.

(b) Find V : COLS. ARE e'VECTORS OF $A^T A$

$$A^T A = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} = \begin{pmatrix} 5 & 20 \\ 20 & 80 \end{pmatrix}$$

$$\underline{\lambda = 0 = \sigma_2^2}: \left(\begin{array}{cc|c} 5 & 20 & 0 \\ 20 & 80 & 0 \end{array} \right) \Rightarrow \left(\begin{array}{cc|c} 5 & 20 & 0 \\ 0 & 0 & 0 \end{array} \right) \Rightarrow x_1 + 4x_2 = 0$$

$$\therefore \begin{pmatrix} -4 \\ 1 \end{pmatrix} \text{ is e'VECTOR CORR. TO } \lambda = 0 \quad \left\| \begin{pmatrix} -4 \\ 1 \end{pmatrix} \right\| = \sqrt{17}$$

$$\therefore v_2 = \begin{pmatrix} -4/\sqrt{17} \\ 1/\sqrt{17} \end{pmatrix}$$

$$\underline{\lambda = 85 = \sigma_1^2}: \left(\begin{array}{cc|c} -80 & 20 & 0 \\ 20 & -5 & 0 \end{array} \right) \Rightarrow \left(\begin{array}{cc|c} -80 & 20 & 0 \\ 0 & 0 & 0 \end{array} \right) \Rightarrow -4x_1 + x_2 = 0$$

$$\therefore \begin{pmatrix} 1 \\ 4 \end{pmatrix} \text{ is e'VECTOR CORR. TO } \lambda = 85$$

$$\therefore v_1 = \begin{pmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{pmatrix}$$

SO $A = U \Sigma V^T$

$$= \underbrace{\begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}}_U \underbrace{\begin{pmatrix} \sqrt{85} & 0 \\ 0 & 0 \end{pmatrix}}_\Sigma \underbrace{\begin{pmatrix} 1/\sqrt{17} & 4/\sqrt{17} \\ -4/\sqrt{17} & 1/\sqrt{17} \end{pmatrix}}_{V^T}$$

(c) Rank $A = 1$

$$\therefore C(A) = \text{SP}(\text{col 1 of } U) = \text{SP} \left\{ \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \right\}$$

$$N(A^T) = \text{SP}(\text{col 2 of } U) = \text{SP} \left\{ \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \right\}$$

$$C(A^T) = R(A) = \text{SP}(\text{col 1 of } V) = \text{SP} \left\{ \begin{pmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{pmatrix} \right\}$$

$$N(A) = \text{SP}(\text{col 2 of } V) = \text{SP} \left\{ \begin{pmatrix} -4/\sqrt{17} \\ 1/\sqrt{17} \end{pmatrix} \right\}$$

⑩ A 2x2 symmetric w/ unit e'vectors u_1 & u_2
A has e'values $\lambda_1=3$ & $\lambda_2=-2$

∴ A diag'ble (distinct e'values)

and if $A=U\Sigma V^T$ is SVD of A, then

A symmetric $\Rightarrow A^T=A$

$\Rightarrow (U\Sigma V^T)^T = (U\Sigma V^T)$

$\Rightarrow V\Sigma^T U^T = U\Sigma V^T$

$\Rightarrow U=V$ since $\Sigma^T=\Sigma$ is invertible
 U^T, V^T both invertible.

∴ $A = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} u_1^T \\ u_2^T \end{pmatrix}$
 $U \quad \Sigma \quad V^T$

is SVD of A.

⑪ (A) Claim: If A has indep't cols, $A^+ = (ATA)^{-1}A^T$

Pf Let $A=U\Sigma V^T$ be the SVD of A

$\begin{pmatrix} m \\ n \end{pmatrix}$

Suppose A is $m \times n \Rightarrow \Sigma$ is also $m \times n = \begin{pmatrix} \sigma_1 & & 0 \\ 0 & \dots & \sigma_n \\ & & 0 \end{pmatrix}$

Since A has LI cols, $m \geq n$.

By defn, $A^+ = V\Sigma^+U^T$ where $\Sigma^+ = \begin{pmatrix} 1/\sigma_1 & & 0 & \\ & \dots & & \\ 0 & & 1/\sigma_n & \\ & & & 0 \end{pmatrix}$.

WTS: $A^+ = (ATA)^{-1}A^T$

Let's compute the RHS using the SVD, and show that it equals $V\Sigma^+U^T$.

$(ATA)^{-1}A^T = [(U\Sigma V^T)^T U \Sigma V^T]^{-1} (U\Sigma V^T)^T$
 $= [V\Sigma^T U^T U \Sigma V^T]^{-1} V\Sigma^T U^T$
 $= [V(\Sigma^T \Sigma) V^T]^{-1} V\Sigma^T U^T$

#19 cont. At this point, note that $\Sigma^T \Sigma$ is diagonal and since A has full col. rank, $\sigma_1 > \dots > \sigma_n > 0$ and \therefore all of the diag entries of $\Sigma^T \Sigma$ are nonzero

$\Rightarrow \Sigma^T \Sigma$ is invertible.

Also, recall that $V^{-1} = V^T$.

$$\begin{aligned} \therefore (A^T A)^{-1} A^T &= (V^T)^{-1} (\Sigma^T \Sigma)^{-1} V^{-1} V^T \Sigma^T U^T \\ &= V (\Sigma^T \Sigma)^{-1} \Sigma^T U^T \end{aligned}$$

So, it remains to show this equals Σ^+ .

$$\Sigma^T \Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n & \\ & & & 0 \end{pmatrix} \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \\ & & & 0 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \\ & & & 0 \end{pmatrix}$$

$$\Rightarrow (\Sigma^T \Sigma)^{-1} = \begin{pmatrix} 1/\sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_n^2 \\ & & & 0 \end{pmatrix}$$

$$\begin{aligned} \Rightarrow (\Sigma^T \Sigma)^{-1} \Sigma^T &= \begin{pmatrix} 1/\sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_n^2 \\ & & & 0 \end{pmatrix} \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \\ & & & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1/\sigma_1 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_n \\ & & & 0 \end{pmatrix} = \Sigma^+ \quad \checkmark \end{aligned}$$

$X^+ \equiv A^+ b$.

Claim: $\left. \begin{array}{l} \textcircled{1} X^+ \in \text{RowSP}(A) \\ \textcircled{2} A^T A X^+ = A^T b \end{array} \right\}$ see back of book for pf.

Ⓐ A has eivals $\lambda = 0, 0, 0, 0, 0$.

Note geo. mult. of $(\lambda=0) = \dim(\text{Nul } A) = 2$.

∴ we'll have 2 Jordan blocks.

The largest block has order 3 since $(A - 0I)^3 = 0$.

• $y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in \text{Nul}(A - 0I)^3$ but $y \notin \text{Nul } A$.

So we form a chain of length 3:

$$Ay = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad A(Ay) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad A(A(Ay)) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

∴ $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is chain of length 3

3 CORR. to 3×3 Jordan block.

• $y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \text{Nul}(A - 0I)^2$ but $y \notin \text{Nul } A$.

$$Ay = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad A(Ay) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

∴ $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is chain of length 2 CORR. to 2×2 Jordan block.

So $A = M^{-1}JM$ where

$$J = \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right), \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



#6, cont.

A good reference for this is

App. B

Solve $\frac{du}{dt} = Au$.

"MATRIX ANALYSIS" by HORN & JOHNSON, pp. 132-134

$A = M^{-1}JM$. Let $u = MV$. Then $\frac{du}{dt} = M \frac{dV}{dt}$.

$\implies M \frac{dV}{dt} = Au = AMV$.

$\implies \boxed{\frac{dV}{dt} = M^{-1}AMV = JV}$ ($v = M^{-1}u$).

Let $v(t) = \begin{pmatrix} v_1(t) \\ \vdots \\ v_5(t) \end{pmatrix}$

Then $\begin{pmatrix} v_1'(t) \\ \vdots \\ v_5'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1(t) \\ \vdots \\ v_5(t) \end{pmatrix} \implies \begin{cases} v_1'(t) = v_2(t) \\ v_2'(t) = v_3(t) \\ v_3'(t) = 0 \\ v_4'(t) = v_5(t) \\ v_5'(t) = 0 \end{cases}$

so $v_5(t) = c_5$

$v_3(t) = c_3$

$v_4(t) = c_5 t + c_4$

$v_2(t) = c_3 t + c_2$

$v_1(t) = c_1 + c_2 t + \frac{c_3}{2} t^2$

$\implies \begin{pmatrix} u_1(t) \\ \vdots \\ u_5(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 + c_2 t + \frac{c_3}{2} t^2 \\ c_3 t + c_2 \\ c_3 \\ c_5 t + c_4 \\ c_5 \end{pmatrix}$

$= \begin{pmatrix} c_1 + c_2 t + \frac{c_3}{2} t^2 \\ (c_3 + c_5)t + c_2 + c_4 \\ c_3 t + c_2 \\ c_3 + c_5 \\ c_3 \end{pmatrix} = u(t)$.