Math 102
Winter '08
Homework # 9

$\boxed{5.5}$

(16.) a) A real & sym $\Rightarrow$ A has all real eigenvalues

b) A stable $\Rightarrow$ $\text{Re}\,\lambda < 0 \quad \forall \lambda \in \Lambda(A)$

c) A orthogonal $\Rightarrow$ $|\lambda| = 1 \quad \forall \lambda \in \Lambda(A)$

d) A Markov $\Rightarrow |\lambda| \leq 1 \quad \forall \lambda \in \Lambda(A)$, and $\lambda = 1$ is an eigenvalue of A.

e) A defective $\Rightarrow$ A has at least one repeated eigenvalue

f) A singular $\Rightarrow$ $0 \in \Lambda(A)$.

(17) U, V unitary

Claim: UV is unitary

Pf WTS: $(UV)^H (UV) = I$

$$(UV)^H (UV) = V^H U^H U V = V^H I V \quad \text{since } U^H U = I$$
$$= V^H V$$
$$= I \quad \text{since } V \text{ unitary} \quad \boxed{\checkmark}$$

(18) (a) U unitary. Let U have eigenvalues $\lambda_1, \ldots, \lambda_n$.

Then $|\lambda_1| = \cdots = |\lambda_n| = 1$ (Property 2')

$$\therefore |\det U| = |\lambda_1 \cdots \lambda_n|$$
$$= |\lambda_1| \cdots |\lambda_n|$$
$$= 1 \cdots 1$$
$$= 1 \quad \checkmark$$

(b) Claim: In general, $\det U \neq \det U^H$

Pf: Just need a counterexample!

Take $U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$

18 cont.

(c) Let $U = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ be unitary.

Then $U^H U = I \Rightarrow \begin{pmatrix} \bar{w} & \bar{y} \\ \bar{x} & \bar{z} \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \bar{w}w + \bar{y}y & \bar{w}x + \bar{y}z \\ \bar{x}w + \bar{z}y & \bar{x}x + \bar{z}z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} &|w|^2 + |y|^2 = 1 \\ &|x|^2 + |z|^2 = 1 \\ &\begin{pmatrix} w \\ y \end{pmatrix}^H \begin{pmatrix} x \\ z \end{pmatrix} = 0 \end{aligned}}$$

(36) $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\sin^2\theta + \cos^2\theta = 1$
$\cos^2\theta - 1 = -\sin^2\theta$

$|Q - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$

$\Leftrightarrow (\cos\theta - \lambda)^2 + \sin^2\theta = 0$

$\Leftrightarrow \underline{\cos^2\theta} - 2\lambda\cos\theta + \lambda^2 + \underline{\sin^2\theta} = 0$

$\Leftrightarrow \lambda^2 - 2\lambda\cos\theta + 1 = 0$

$\Leftrightarrow \lambda = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \dfrac{2\cos\theta \pm 2\sqrt{\cos^2\theta - 1}}{\cancel{2}}$

$\Leftrightarrow \lambda = \cos\theta \pm i\sin\theta$

$\Leftrightarrow \lambda = \cos\theta + i\sin\theta \boxed{= e^{i\theta}}$

OR $\lambda = \cos\theta - i\sin\theta = \cos\theta + i\sin(-\theta) \boxed{= e^{-i\theta}}$

E'vec. corr. to $\lambda = e^{i\theta}$ is $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

E'vec. corr. to $\lambda = e^{-i\theta}$ is $\begin{pmatrix} 1 \\ i \end{pmatrix}$

(38) $v_1, \dots, v_n$ o.n. basis for $\mathbb{C}^n \Rightarrow$ matrix with those columns is an orthogonal matrix

Let $z \in \mathbb{C}^n$.

Claim $z = (v_1^H z)v_1 + \cdots + (v_n^H z)v_n$.

Pf $z \in \mathbb{C}_n$, $\{v_1, \dots, v_n\}$ basis for $\mathbb{C}^n \Rightarrow$

$z = \alpha_1 v_1 + \cdots + \alpha_n v_n$ for some $\alpha_i \in \mathbb{C}^n$

$$\therefore v_1^H z = v_1^H \alpha_1 v_1 + \cdots + v_1^H \alpha_n v_n$$
$$= \alpha_1 (v_1^H v_1) + \cdots + \alpha_n \cancel{(v_1^H v_n)} \to 0$$
$$= \alpha_1 \quad \text{since} \quad v_1^H v_1 = 1 \text{ and } v_1^H v_j = 0 \;\forall j \neq 1$$

$\therefore v_1^H z = \alpha_1$

Similarly, $\alpha_i = v_i^H z \;\forall i$ $\boxed{\checkmark}$

(39) $\int_0^{2\pi} \overline{e^{-ix}} \cdot e^{ix}\, dx = 0$.

(41) A Hermitian $\Rightarrow (R+iS)^H = R+iS$

$\Rightarrow R^H - iS^H = R + iS$

$\Rightarrow R^H = R,\; S^H = -S$

$\Rightarrow R$ symmetric & S skew symmetric.

(42) The (complex) dimension of $\mathbb{C}^n$ is $\boxed{n}$

Nonreal basis for $\mathbb{C}^n$: $\underbrace{\left\{ \begin{pmatrix} i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ i \end{pmatrix} \right\}}_{n \text{ vectors}}$

(44) Let $\lambda \in \Lambda(A)$. Then for some $x \neq 0$,

$$Ax = \lambda x$$

$$\Rightarrow (Ax)^H = (\lambda x)^H$$

$$\Rightarrow x^H A^H = \lambda^H x^H$$

$$\Rightarrow x^H A^H = \bar{\lambda} x^H$$

so eigenvalues of A^H are complex conjugates of eigenvalues of A.

(46) $A + iB$ unitary with A, B real.

Claim $Q = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ is orthogonal.

Pf WTS: $Q^H Q = I$

$A + iB$ unitary $\Rightarrow (A+iB)^H (A+iB) = I$

$$\Rightarrow (A^H - iB^H)(A + iB) = I$$

$$\Rightarrow A^H A + iA^H B - iB^H A + B^H B = I + i0$$

$$\Rightarrow (A^H A + B^H B) + i(A^H B - B^H A) = I + i0$$

Equating real & imaginary parts gives us

$$A^H A + B^H B = I$$

$$A^H B - B^H A = 0 \Rightarrow A^H B = B^H A.$$

Now, $Q^H Q = \begin{pmatrix} A^H & B^H \\ -B^H & A^H \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} A^H A + B^H B & -A^H B + B^H A \\ -B^H A + A^H B & B^H B + A^H A \end{pmatrix}$

$$= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = I$$

$\boxed{\checkmark}$

$\boxed{5.6}$

(3) $\lambda \in \Lambda(A) \Rightarrow Ax = \lambda x$ for some $x \neq 0$

$\Rightarrow (A-I)x = Ax - x = \lambda x - x = (\lambda - 1)x$

$\therefore \lambda - 1 \in \Lambda(A-I)$.

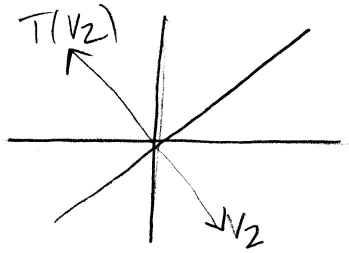
$\therefore A, A-I$ don't have the same e'values

$\Rightarrow A, A-I$ not similar.

(8) $V = \left\{ \underset{v_1}{\begin{pmatrix}1\\1\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}1\\4\end{pmatrix}} \right\}$ $\quad W = \left\{ \underset{w_1}{\begin{pmatrix}2\\5\end{pmatrix}}, \underset{w_2}{\begin{pmatrix}1\\4\end{pmatrix}} \right\}$

$$\left.\begin{array}{l} v_1 = w_1 - w_2 \Rightarrow [v_1]_W = \begin{pmatrix}1\\-1\end{pmatrix} \\ v_2 = 0w_1 + w_2 \Rightarrow [v_2]_W = \begin{pmatrix}0\\1\end{pmatrix} \end{array}\right\} \underset{W \leftarrow V}{M} = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.$$

(11)



$V = \left\{ \underset{v_1}{\begin{pmatrix}1\\1\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}1\\-1\end{pmatrix}} \right\}$ $\quad W = \left\{ \underset{w_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \underset{w_2}{\begin{pmatrix}0\\1\end{pmatrix}} \right\}$

$T(v_1) = v_1 = 1v_1 + 0v_2$

$T(v_2) = \begin{pmatrix}-1\\1\end{pmatrix} = 0v_1 - 1v_2$

$\therefore T_V = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$T(w_1) = w_2$

$T(w_2) = w_1$

$\therefore T_W = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

<u>WTS</u>: T_V and T_W are similar.

<u>Pf</u> let $M = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ (M changes basis from V to W)

Then $M T_V M^{-1} = T_W$.

$\therefore T_V$ and T_W are similar.

(13) (A) P_2 has basis $\{1, x, x^2\} = \mathcal{B}$

$$\frac{d}{dx}1 = 0 \Rightarrow [T(1)]_{\mathcal{B}} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

$$\frac{d}{dx}x = 1 \Rightarrow [T(x)]_{\mathcal{B}} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\frac{d}{dx}x^2 = 2x \Rightarrow [T(x^2)]_{\mathcal{B}} = \begin{pmatrix}0\\2\\0\end{pmatrix}$$

$$\therefore D = \begin{pmatrix}0&1&0\\0&0&2\\0&0&0\end{pmatrix}$$

Check $\begin{pmatrix}0&1&0\\0&0&2\\0&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}b\\2c\\0\end{pmatrix}$ ✓

(B) $$D^3 = \begin{pmatrix}0&1&0\\0&0&2\\0&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&2\\0&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&2\\0&0&0\end{pmatrix}$$

$$= \begin{pmatrix}0&1&0\\0&0&2\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&2\\0&0&0\\0&0&0\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

Interpretation: Third derivative of a degree-2 poly. is always zero.

(C) eigenvalues all 0.
eigenvectors: $\text{sp}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$ (corresponding to constant polys).

Note: This means D is not diag'ble!

(17) (A) <u>WTS</u>: $TT^H = I$

$TT^H = (U^{-1}AU)(U^{-1}AU)^H$

$= U^{-1}A\underbrace{\cancel{UU^H}}_{I} A^H (U^{-1})^H$ since U unitary

$= U^{-1}AA^H(U^{-1})^H$

$= U^{-1}(U^{-1})^H$ since A unitary

$= U^H \cdot (U^H)^H$ since U unitary $\Rightarrow U^H = U^{-1}$

$= U^H U$

$= I$ since U unitary. ✓

(B) Let T be unitary & upper Δ.

<u>WTS</u>: T diagonal.

T upper $\Delta \Rightarrow$ diag entries are eigenvalues of T

$\Rightarrow |t_{ii}| = 1 \quad \forall \quad 1 \leq i \leq n$.

But T unitary $\Rightarrow$ columns all have norm 1.

$\therefore$ off-diagonal entries all <u>zero</u>.

$\therefore$ T is diagonal. ✓

(25) Just plug in $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(30) (a) $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ Find M s.t. $B = M^{-1}AM$.

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{M^{-1}}\underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{M} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}}_{B}.$$

(b) $$\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{M^{-1}}\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{M} = \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{B}$$

30 cont.

ⓒ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$

$\quad M^{-1} \qquad A \qquad M \qquad\quad B$

(31) See back of book.

(41) ⓐ True: A singular $\Rightarrow 0 \in \Lambda(A)$

B inv'ble $\Rightarrow 0 \notin \Lambda(B)$

$\therefore$ A not similar to B.

ⓑ False A diag'ble nonsymmetric matrix is necessarily similar to a diagonal matrix, which is always symmetric!

ⓒ False: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

ⓓ True: (same idea as #3).

(42) Claim: AB has same e'values as BA

Pf: Assume $\lambda \neq 0$ and $\lambda \in \Lambda(AB)$.

Then $ABx = \lambda x$ for some $x \neq 0$

$\Rightarrow B(ABx) = B(\lambda x)$

$\Rightarrow BA(Bx) = \lambda(Bx)$

$\therefore$ λ is an e'val of BA with corr. e'vector Bx

Note $Bx \neq 0$ since if $Bx = 0$, $ABx = 0 \Rightarrow \lambda = 0$, but we have assumed $\lambda \neq 0$.

$\therefore$ If $\lambda \neq 0$ is an e'val of AB, then it is also an e'val of BA

$\longrightarrow$

42, cont.

Now suppose $\lambda \in \lambda(BA)$ (again, $\lambda \neq 0$).
Then $BAx = \lambda x$ for some $x \neq 0$

$\Rightarrow ABAx = A(\lambda x)$

$\Rightarrow AB(Ax) = \lambda(Ax)$

$\Rightarrow \lambda$ is e'val. of AB (with corr. e'vector Ax).

Note $Ax \neq 0$ since if $Ax = 0$ then $\lambda = 0$, but we have assumed $\lambda \neq 0$.

Last case: must show

$$0 \in \lambda(AB) \Leftrightarrow 0 \in \lambda(BA).$$

Recall that a matrix has a zero eigenvalue iff it is singular (has det = 0). It is easy to see that $\det(AB) = 0 \Leftrightarrow \det(BA) = 0$, so we have proven the case where $\lambda = 0$. ☑

44. $A \sim B \Rightarrow B = M^{-1}AM$

(a) $\Rightarrow B^2 = (M^{-1}AM)(M^{-1}AM) = M^{-1}A^2M$

$\Rightarrow B^2 \sim A^2$.

(b) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Then A is not similar to B since $M^{-1}BM = 0 \ \forall M$

but $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = B^2$

so A^2 is (trivially) similar to B^2!

(c) Let $M = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Then $B = M^{-1}AM$.

(d) Assume that $B = M^{-1}AM \Rightarrow B = M^{-1}(3I)M = 3(M^{-1}IM) = 3I$.
This is a ↯ b/c $B \neq 3I$. ∴ B and A are not similar.

(e) The prescribed action is a similarity transformation! (see #30 (c)).